Elektromagnetika, kolokvij 12.12.2000, rešitve

1) $\rho(r,\theta,\phi) = \ ? \quad \frac{\partial}{\partial\phi} = 0$

$V(r,\theta,\phi) = V_0\, r \sin 3\theta$

$\vec{E} = -\,grad\ V = \underline{-\vec{1}_r\ V_0 \sin 3\theta - \vec{1}_\theta\ V_0\, 3\cos 3\theta}$

$\rho = div\ \vec{D} = div\ \varepsilon_0\,\vec{E} = \varepsilon_0 \frac{1}{r^2 \sin\theta}\left[-\frac{\partial}{\partial r}\left(r^2 \sin\theta\ V_0 \sin 3\theta\right) - \frac{\partial}{\partial\theta}\left(r \sin\theta\ V_0\, 3\cos 3\theta\right)\right] =$

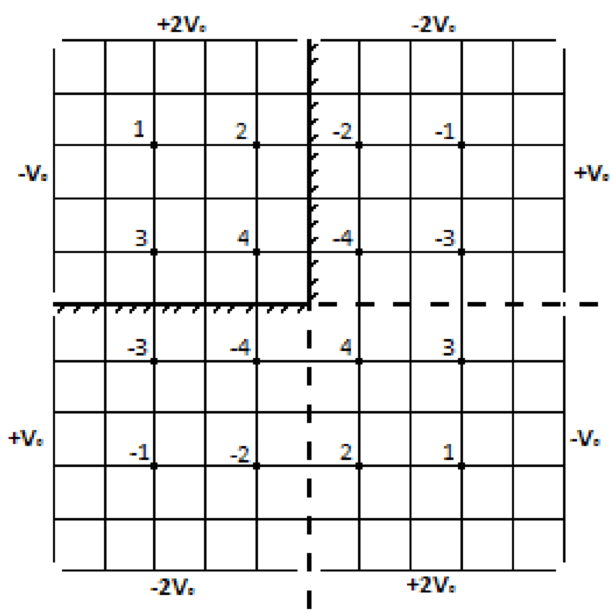
$= \varepsilon_0 \frac{1}{r^2 \sin\theta}\left[-V_0\, 2r \sin\theta \sin 3\theta - 3V_0\, r\left(\cos\theta\cos 3\theta + \sin\theta\,(-3\sin 3\theta)\right)\right] =$

$= \varepsilon_0 \frac{V_0}{r \sin\theta}\left[-2\sin\theta\sin 3\theta - 3\cos\theta\cos 3\theta + 9\sin\theta\sin 3\theta\right]$

$= \varepsilon_0 \frac{V_0}{r}\left(7\sin 3\theta - 3\,\mathrm{ctg}\,\theta\cos 3\theta\right)$

---

2)



$4V_1 = -V_0 + 2V_0 + V_2 + V_3 \quad \rightarrow \quad 4V_1 = V_0 + V_2 + V_3$

$4V_2 = 2V_0 + V_1 - V_2 + V_4 \quad \rightarrow \quad 5V_2 = 2V_0 + V_1 + V_4$

$4V_3 = -V_0 + V_1 - V_3 + V_4 \quad \rightarrow \quad 5V_3 = -V_0 + V_1 + V_4$

$4V_4 = V_2 + V_3 - V_4 - V_4 \quad \rightarrow \quad 6V_4 = V_2 + V_3 \quad \rightarrow \quad V_4 = \frac{V_2 + V_3}{6}$

$19V_2 = 9V_0 + V_3 + 4V_4 \quad \rightarrow \quad 55V_2 = 27V_0 + 5V_3$

$19V_3 = -3V_0 + V_2 + 4V_4 \quad \rightarrow \quad 55V_3 = -9V_0 + 5V_2$

$V_2 = \frac{12}{25}V_0$

$V_3 = -\frac{3}{25}V_0$

$V_4 = \frac{3}{50}V_0$

$V_1 = \frac{17}{50}V_0$

---

3) $V\left(x,y,z = \frac{c}{2}\right) = V_0\, sin\left(\frac{\pi}{a}\,x\right) \qquad V\left(x,y,z = -\frac{c}{2}\right) = -V_0\, sin\left(\frac{\pi}{a}\,x\right)$

$\frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y} \neq 0 \quad \frac{\partial}{\partial z} \neq 0 \qquad k_x = \frac{\pi}{a}, \quad k_y = \frac{\pi}{b}, \qquad k_z = \sqrt{{k_x}^2 + {k_y}^2} = \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2},$

Nastavek: $V\left(x,y,z = \frac{c}{2}\right) = \sum_m \sum_n C_{mn}\, sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\ \frac{c}{2}\right) = V_0\ sin\ \frac{\pi}{a}$

$sinh\left(k_z\,(-\frac{c}{2})\right) = -\,sinh\left(k_z\,\frac{c}{2}\right)$ → rešitev pri $z = -\frac{c}{2}$ glede na vzbujanje $(-\,V_0\ sin\,\frac{\pi}{a}x)$ predstavlja zrcalno sliko glede na rešitev pri $z = \frac{c}{2}$, zato je nesmiselna.

Glede na vsiljen potencial $(sin\,\frac{\pi}{a}x)$ na pokrovih sklepamo: $m = 1$;

$$V_0 = \sum_n C_n\ sin\left(\frac{n\pi}{b}y\right) sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\ \frac{c}{2}\right)$$

Koeficiente C_n določimo s Fourierjevo analizo, tako da čimbolje zadostimo pogoju $V = V_0$ na pokrovu:

$$\int_0^b V_0\left(sin\left(\frac{l\pi}{b}y\right)\right)dy = \int_0^b C_n\ sin\left(\frac{n\pi}{b}y\right) sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\ \frac{c}{2}\right) sin\left(\frac{l\pi}{b}y\right)dy\ ; \qquad n \to l$$

$$\frac{V_0\,b}{l\pi}\,(1-\cos(l\pi)) = C_l\ sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\ \frac{c}{2}\right)\left(\frac{b}{2}\right)$$

$$C_l = \frac{2\,V_0}{l\pi\,sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{l\pi}{b}\right)^2}\ \frac{c}{2}\right)}(1-\cos(l\pi))$$

$$V(x,y,z) = \sum_{l=1}^{\infty}\frac{2\,V_0}{l\pi}(1-\cos(l\pi))\ sin\left(\frac{\pi}{a}x\right) sin\left(\frac{l\pi}{b}y\right)\frac{sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\ z\right)}{sinh\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2}\ \frac{c}{2}\right)}$$

---

4) $\frac{\partial}{\theta r} \neq 0 \quad \frac{\partial}{\partial\theta} \neq 0 \quad \frac{\partial}{\partial\varphi} = 0 \qquad \vec{J}_0 = -\vec{1}_z J_0 = -\vec{1}_z\gamma_0 E_0$

$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0 \qquad \vec{E} = -grad\ V = -\vec{1}_z\,\frac{\partial V}{\partial z} \quad \to \quad \frac{\partial V_0}{\partial z} = E_0\ \int dz$

V prostoru "γ_0": Potencial: $V_0\ (\mathrm{r},\theta) = (A_0 r + B_0 r^{-2})\ cos\theta$

Polje: $\vec{E}_0 = -grad\ V = -\vec{1}_r\,(A_0 - \frac{2B_0}{r^3})\ cos\theta + \vec{1}_\theta\,(A_0 + \frac{B_0}{r^3})\ sin\theta$

V prostoru "γ_N": Potencial: $V_N\ (\mathrm{r},\theta) = (A_N r + B_N r^{-2})\ cos\theta$

Polje: $\vec{E}_N = -grad\ V = -\vec{1}_r\,(A_N - \frac{2B_N}{r^3})\ cos\theta + \vec{1}_\theta\,(A_N + \frac{B_N}{r^3})\ sin\theta$

1. Pri $r \gg b$ je polje nespremenjeno:

   $\vec{E}_0 = -\vec{1}_r\,(A_0 - \frac{2B_0}{r^3})\ cos\theta + \vec{1}_\theta\,(A_0 + \frac{B_0}{r^3})\ sin\theta \approx -\vec{1}_r\,A_0\ cos\theta + \vec{1}_\theta\,A_0\ sin\theta = -\vec{1}_z\,A_0 = -\vec{1}_z\,E_0 \ \to\ \underline{A_0 = E_0}$

2. Pri $r = a$ (na površini krogle) je: $\vec{E}_t = \vec{E}_\theta = 0 \quad ; \quad \vec{1}_\theta E_N(r=a) = 0 = (A_N + \frac{B_N}{r^3})\ sin\theta \to B_N = -A_N a^3$

3. Pri $r = b$ je: $\vec{E}_{tN} = \vec{E}_{t0} \quad ; \quad \vec{J}_{nN} = \vec{J}_{n0}$

   $A_N + \frac{B_N}{b^3} = E_0 + \frac{B_0}{b^3} \quad ; \quad \gamma_N(A_N - \frac{2B_N}{b^3}) = \gamma_0(E_0 - \frac{2B_0}{b^3})$

   $A_N(1 - \frac{a^3}{b^3}) = E_0 + \frac{B_0}{b^3} \quad ; \quad A_N\gamma_N(1 + \frac{2a^3}{b^3}) = E_0\gamma_0 - \frac{2B_0}{b^3}\gamma_0$

   $A_N(1 - \frac{a^3}{b^3}) = E_0 + \frac{B_0}{b^3} \quad ; \quad E_0\gamma_N b^3\left(1 + \frac{2a^3}{b^3}\right) + B_0\gamma_N\left(1 + \frac{2a^3}{b^3}\right) = E_0\gamma_0 b^3\left(1 - \frac{a^3}{b^3}\right) - 2B_0\gamma_0\left(1 - \frac{a^3}{b^3}\right)$

$$A_N = \frac{E_0 b^3 + B_0}{b^3\left(1-\frac{a^3}{b^3}\right)} \qquad ; \qquad B_0 = \frac{E_0 b^3\left(\gamma_0\left(1-\frac{a^3}{b^3}\right)-\gamma_N\left(1+\frac{2a^3}{b^3}\right)\right)}{2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)}$$

$$A_N = \frac{3E_0\gamma_0}{2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)} \qquad ; \qquad B_N = -\frac{3E_0\gamma_0 a^3}{2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)}$$

$$\vec{E}_0 = -\vec{1}_r\, E_0 \left(1 - \frac{b^3\left(\gamma_0\left(1-\frac{a^3}{b^3}\right)-\gamma_N\left(1+\frac{2a^3}{b^3}\right)\right)}{r^3\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)}\right) cos\theta + \vec{1}_\theta\, E_0 \left(1 + \frac{b^3\left(\gamma_0\left(1-\frac{a^3}{b^3}\right)-\gamma_N\left(1+\frac{2a^3}{b^3}\right)\right)}{r^3 2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)}\right) sin\theta$$

$$\vec{E}_N = -\vec{1}_r\, \frac{3E_0\gamma_0}{2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)} \left(1 - \frac{2a^3}{r^3}\right) cos\theta + \vec{1}_\theta\, \frac{3E_0\gamma_0}{2\gamma_0\left(1-\frac{a^3}{b^3}\right)+\gamma_N\left(1+\frac{2a^3}{b^3}\right)} \left(1 + \frac{a^3}{r^3}\right) sin\theta$$

---

5) $\vec{F} = \vec{1}_\rho + \vec{1}_\varphi\, \rho + \vec{1}_z\, \rho\cos\varphi$

$$\oint \vec{F}\cdot d\vec{s} = \int_2^4 \vec{F}\ \vec{1}_z\, dz + \int_\pi^{-\frac{\pi}{2}} \vec{F}\left(-\vec{1}_\varphi\right)\rho(-d\varphi) + \int_2^1 \vec{F}\ \left(-\vec{1}_\rho\right)(-d\rho) + \int_4^2 \vec{F}\ \left(-\vec{1}_z\right)(-dz) + \int_1^2 \vec{F}\ \vec{1}_\rho\, d\rho +$$

$$+ \int_{-\frac{\pi}{2}}^{\pi} \vec{F}\ \vec{1}_\varphi\, \rho\, d\varphi =$$

$$= \Big|_{\substack{\rho=2\\ \varphi=\pi}} \rho\cos\varphi\ \Big[_2^4\, z + \Big|_{\substack{\rho=2\\ z=4}} \rho^2\ \Big[_{\pi}^{-\frac{\pi}{2}}\, \varphi + \Big|_{\substack{\varphi=-\pi/2\\ z=4}} \Big[_2^1\, \rho + \Big|_{\substack{\rho=1\\ \varphi=-\pi/2}} \rho\cos\varphi\ \Big[_4^2\, z + \Big|_{\substack{\varphi=-\pi/2\\ z=2}} \Big[_1^2\, \rho + \Big|_{\substack{\rho=2\\ z=2}} \rho^2\ \Big[_{-\frac{\pi}{2}}^{\pi}\, \varphi =$$

$$= 2\,(-1)\,2 + 4\left(-\frac{\pi}{2} - \pi\right) + (1-2) + 0 + (2-1) + 4\left(\pi + \frac{\pi}{2}\right) = -4$$

Elektromagnetika, kolokvij 15.02.2001, rešitve

1) $\vec{J} = \vec{1}_y J_0 \quad [A/m], \qquad r \gg a$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{v'} \vec{J}(\vec{r}')\frac{e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dv' = \frac{\mu_0 J_0}{4\pi}\int_{v'} \vec{1}_y \frac{e^{-jkr}}{r}dv' = \frac{\mu_0 a^3}{4\pi} J_0 \frac{e^{-jkr}}{r}\left(\vec{1}_r \sin\theta\sin\phi + \vec{1}_\theta \cos\theta\sin\phi + \vec{1}_\phi \cos\phi\right)$$

$$\vec{H} = \frac{1}{\mu}\, rot\, \vec{A} = \frac{a^3}{4\pi} J_0 \frac{1}{r^2 \sin\theta}\begin{vmatrix} \vec{1}_r & r\vec{1}_\theta & r\sin\theta\,\vec{1}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \sin\theta\sin\phi\frac{e^{-jkr}}{r} & r\cos\theta\sin\phi\frac{e^{-jkr}}{r} & r\sin\theta\cos\phi\frac{e^{-jkr}}{r} \end{vmatrix} =$$

$$= \frac{a^3}{4\pi} J_0 \frac{1}{r^2\sin\theta}\Big[\vec{1}_r\left(\cos\theta\cos\phi\, e^{-jkr} - \cos\theta\cos\phi\, e^{-jkr}\right) + r\,\vec{1}_\theta\left(\sin\theta\cos\phi\,\frac{e^{-jkr}}{r} + (jk)\sin\theta\cos\phi\, e^{-jkr}\right) +$$

$$+ r\sin\theta\;\vec{1}_\phi\left((-jk)\cos\theta\sin\phi\, e^{-jkr} - \cos\theta\sin\phi\,\frac{e^{-jkr}}{r}\right)\Big]$$

$$\vec{H} = \frac{a^3}{4\pi} J_0 \frac{e^{-jkr}}{r}\left[\left(\frac{1}{r} + jk\right)\left(\vec{1}_\theta\cos\phi - \vec{1}_\phi\cos\theta\sin\phi\right)\right]$$

---

2) $\vec{K}(z=0) = \vec{1}_\rho K_0\, J_0(\alpha\rho) \qquad \sigma = ?$

$$\sigma = \frac{j}{\omega} div\,\vec{K} = \frac{j}{\omega}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\,\rho\;K_0\,J_0(\alpha\rho)\right) = \frac{j}{\omega}\frac{K_0}{\rho}\left(J_0(\alpha\rho) + \rho\alpha\, J_0'(\alpha\rho)\right)$$

---

3) $a = 4\ cm, \quad b = 3\ cm, \quad c = 9\ cm, \qquad \varepsilon_r = ? \qquad f_{l,m,n} = \frac{c_0}{2}\sqrt{\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{n}{c}\right)^2}$ m,n,p → cela števila

$f_{011} - f_{101} = 288{,}8\ MHz \qquad \frac{c_0}{2\sqrt{\varepsilon_r}}\left(\sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2} - \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{c}\right)^2}\right) = 288{,}8\ MHz$

$$\varepsilon_r = \left(\frac{c_0}{2\ x\ 288{,}8\ MHz}\left(\sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2} - \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{c}\right)^2}\right)\right)^2 = 16{,}32$$

---

4) $rot\,\vec{H} = \gamma\vec{E} + j\omega\varepsilon\vec{E} = j\omega\varepsilon_0\underbrace{\left(\frac{\gamma}{j\omega\varepsilon_0} + 1\right)}_{\varepsilon_r}\vec{E}$

$k = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}$ } V praznem prostoru

$$k = k_0\sqrt{1 + \frac{\gamma}{j\omega\varepsilon_0}} \approx k_0\left(1 + \frac{\gamma}{2j\omega\varepsilon_0}\right) = k_0\left(1 - j\frac{\gamma}{2\omega\varepsilon_0}\right) = \beta_0 - j\alpha \quad \rightarrow \quad \alpha = \frac{\gamma}{2\omega\varepsilon_0}$$

$$l' = l\,\frac{k}{\beta} = \frac{\sqrt{(\beta)^2 + \left(\frac{\pi}{b}\right)^2}}{\beta} \quad \rightarrow \quad \alpha' = \alpha\,\frac{\sqrt{(\beta)^2 + \left(\frac{\pi}{b}\right)^2}}{\beta} \qquad \alpha_{dB/m} = \alpha'_{Np/m}\,\frac{20}{\ln 10}$$

5) $\vec{1}_{kv} = \vec{1}_z$ $\quad$ $\vec{1}_{ko} = -\vec{1}_z$

$$\vec{E} = \vec{E}_v + \vec{E}_o = \vec{1}_x A e^{-jk_0 z} + j\,\vec{1}_y A e^{-jk_0 z} + \vec{1}_x \Gamma A e^{+jk_0 z} + j\,\vec{1}_y \Gamma A e^{+jk_0 z}$$

$$\vec{H} = \vec{H}_v + \vec{H}_o = \left(\vec{1}_{kv} \times \frac{\vec{E}_v}{Z}\right) + \left(\vec{1}_{ko} \times \frac{\vec{E}_o}{Z}\right) = \vec{1}_y \frac{A}{Z} e^{-jk_0 z} - j\,\vec{1}_x \frac{A}{Z} e^{-jk_0 z} - \vec{1}_y \frac{\Gamma A}{Z} e^{+jk_0 z} + j\,\vec{1}_x \frac{\Gamma A}{Z} e^{+jk_0 z}$$

$$S = \frac{1}{2}\left|\vec{E} \times \vec{H}^*\right| = \frac{1}{2}\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ Ae^{-jk_0 z} + \Gamma A e^{+jk_0 z} & j\,Ae^{-jk_0 z} + j\,\Gamma A e^{+jk_0 z} & 0 \\ j\,\frac{A^*}{Z} e^{+jk_0 z} - j\,\frac{\Gamma^* A^*}{Z} e^{-jk_0 z} & \frac{A^*}{Z} e^{+jk_0 z} - \frac{\Gamma^* A^*}{Z} e^{-jk_0 z} & 0 \end{vmatrix} =$$

$$= \vec{1}_z \left(\frac{AA^*}{Z} - \frac{AA^*\Gamma^*}{Z} e^{-2jk_0 z} + \frac{AA^*\Gamma}{Z} e^{+2jk_0 z} - \frac{AA^*\Gamma\Gamma^*}{Z}\right) = \vec{1}_z \frac{AA^*}{Z}\left(1 - \Gamma^* e^{-2jk_0 z} + \Gamma e^{+2jk_0 z} - \Gamma\Gamma^*\right) =$$

$$= \vec{1}_z \frac{|A|^2}{Z}\left(1 - \Gamma 2 j sin 2k_0 z - |\Gamma|^2\right)$$

Odbojnost:

$$E_v + E_o = E_p \;;\quad -\frac{1}{Z_0}\left[E_v - E_o\right] = -\frac{1}{Z} E_p \;;\qquad \frac{E_o}{E_v} = \frac{Z - Z_0}{Z + Z_0} = \frac{1 - \sqrt{\varepsilon_r}}{1 + \sqrt{\varepsilon_r}} = \Gamma \;;\quad \Gamma^2 = |\Gamma|^2$$

$$\Gamma = \Gamma^* = -|\Gamma| = \frac{\Gamma A e^{+jk_0 z}}{A e^{-jk_0 z}} = \Gamma e^{+2jk_0 z}$$
