An Analytical Model for the Power Spectral Density of a Voltage-Controlled Oscillator and Its Analogy to the Laser Linewidth Theory

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***Abstract*—We calculate the output power density spectrum for a simple voltage-controlled oscillator (VCO) circuit. The power spectral density of the oscillator is composed of a term related to the high-frequency fluctuations in the feedback loop and a term related to the low-frequency fluctuations of the frequency control voltage. The latter is treated stochastically in a similar fashion to the inhomogeneous line broadening of gas lasers due to the Doppler effect. This additional broadening causes a deviation of the power spectral density from the Lorentzian shape, that is, from the phase noise decay of -6 dB per octave. This is specially pronounced at not-too-large frequency offsets. The analogy between electrical oscillators and optical oscillators (lasers) allows the methods used in optical spectroscopy to be applied. The approach may be employed to synthesize oscillator spectra from the equivalent circuit parameters with small numerical effort. Furthermore, it allows experimental power density spectra to be decomposed into the contributions stemming from the high-frequency noise in the feedback loop and the low-frequency noise of the oscillation frequency. This should give better insight into the origin of the phase noise. Besides VCO's, this concept may be useful for oscillators subject to Gaussian supply and substrate noise.**

***Index Terms*—FM noise, oscillator noise, phase noise, voltage-controlled oscillators.**

I. Introduction

VOLTAGE-CONTROLLED oscillators (VCO's) are critical components of modern communication systems. In particular, critical parameters of phase-locked loops, such as spectral purity and power dissipation, strongly depend on the VCO performance. The demands of international standards such as GSM make low-phase-noise, low-power VCO's a topic of current interest [1]–[3]. Accurate modeling of the phase noise of VCO's requires substantial numerical effort [4]. Therefore, a simple analytical model is desirable to facilitate synthesis and analysis of the power spectral density of VCO's. This would also give insight into the dominant noise mechanisms in the circuit.

There is a strong analogy between optical resonators (lasers) and electrical oscillators as outlined below, whereby the vector potential in a laser corresponds to the voltage in an electrical oscillator. The aim of this paper is to analytically calculate the oscillator line broadening by using the methods employed in optical spectroscopy. The power spectral density is expressed in terms of the oscillation frequency, the quality factor of the passive resonator, the power consumed by the load, the equivalent noise resistance in the feedback loop, the load resistance, and the root-mean-square value of the oscillation frequency or of the frequency control voltage, respectively. We discuss the influence of these parameters on the oscillator linewidth and on the relative phase noise.

This paper does not intend to give an overview of the vast number of papers on oscillator phase noise. (For an overview see [4].) Instead it focuses on how the Gaussian noise of the control voltage may be incorporated into the phase noise calculations. To the best of the author's knowledge, this problem has not yet been addressed so far. The algorithm derived here may also be applied to oscillators subject to Gaussian supply and substrate noise, since such oscillators can be considered as VCO's with purely stochastic control voltage.

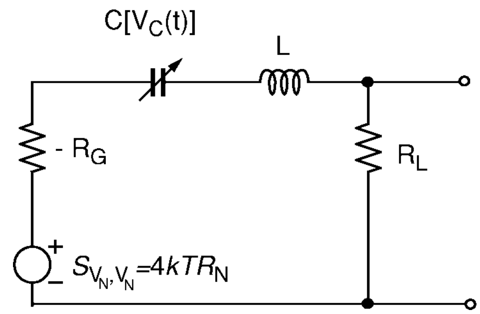

Fig. 1. VCO model. The oscillation frequency is controlled by a fluctuating voltage $V_C(t)$.

II. The Oscillator Model

The passive resonator is modeled by a resistance R_L describing the loss, a tunable capacitance $C(V_C)$ with V_C the frequency control voltage, and an inductance L, which gives the impedance

$$Z_L = R_L + \frac{1}{j\omega C} + j\omega L \tag{1}$$

as shown in Fig. 1. We model the active device by a real negative resistance $-R_G$. The noise in the feedback loop is described by the voltage source v_N in series with $-R_G$ and Z_L. The spectral density S_{v_N,v_N} is expressed by an equivalent

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noise resistance R_N defined by

$$S_{v_N, v_N} = 4kTR_N. \tag{2}$$

The resonance frequency ω_0 of the passive resonator is controlled by the voltage V_C, which fluctuates. These fluctuations give rise to fluctuations of the oscillation frequency ω_0.

So far, several approximations have been made. First, the equivalent circuit for the passive resonator is relatively simple, although it may be generalized by using a more complex algebraic expression for Z_L. Second, the imaginary part of the generator impedance is disregarded. Inclusion of the imaginary part would, as in [5], shift the oscillation frequency, but leave the linewidth unchanged. Third, the frequency dependence of R_N and R_G has been neglected. This is a good approximation, if the curvature of the gain (here described by R_G) as a function of frequency is much smaller at the frequency of oscillation than that of the passive resonator [5]. For typical quality factors of passive electrical oscillators, this condition is easily fulfilled.

III. The Fixed Control Voltage Case

First, we neglect the fluctuations of V_C. The power spectral density S_{v_L, v_L} of the voltage over the load R_L is the Fourier transform of the steady-state autocorrelation function, which can be measured with a spectrum analyzer. According to Fig. 1, the spectral density of the noise voltage over R_L is given by

$$S_{v_L, v_L} = S_{v_N, v_N} \frac{R_L^2}{|Z_L - R_G|^2}. \tag{3}$$

Here, we have exploited the fact that the noise in a narrow frequency range may be treated in the same fashion as in the case of a small-signal analysis. After inserting Z_L we find

$$S_{v_L, v_L} = S_{v_N, v_N} \frac{1}{(1-r_G)^2 + Q^2(\omega/\omega_0 - \omega_0/\omega)^2} \tag{4}$$

where

$$\omega_0 = 1/\sqrt{LC} \tag{5}$$

is the oscillation frequency

$$Q = \frac{\omega_0 L}{R_L} \tag{6}$$

is the quality factor of the passive resonator, and

$$r_G = R_G/R_L \lesssim 1 \tag{7}$$

is the relative gain. As for lasers [5], the gain (here R_G) must be slightly smaller than the total loss (here R_L), otherwise the power spectral density would diverge. As evident from (4), the spectral density becomes very large at the oscillation frequency if R_G approaches R_L. The narrow-bandwidth oscillation may be considered a frequency selective amplification of fluctuations. In the case of electrical oscillators, the fluctuations stem from, e.g., thermal motion of the electrons, and in the case of lasers from spontaneous emission of light. Since we consider only positive frequencies, we may focus on frequencies near ω_0. With the resonance approximation

$$\begin{aligned} \omega/\omega_0 - \omega_0/\omega &= \frac{1}{\omega\omega_0}(\omega+\omega_0)(\omega-\omega_0) \\ &\approx \frac{2}{\omega_0}(\omega-\omega_0) \end{aligned} \tag{8}$$

we find

$$S_{v_L, v_L} \approx S_{v_N, v_N} \frac{1}{(1-r_G)^2 + \left(\frac{2Q}{\omega_0}\right)^2 (\omega-\omega_0)^2}. \tag{9}$$

This expression represents a Lorentz function corresponding to an exponential decay of the autocorrelation function in the time domain. The Lorentzian linewidth γ (half-width at half-maximum) follows from (9) as

$$\gamma = (1-r_G)\frac{\omega_0}{2Q} \tag{10}$$

and the spectrum may be written as

$$S_{v_L, v_L} = S_{v_N, v_N} \left(\frac{\omega_0}{2Q}\right)^2 \frac{\pi}{\gamma} L[\gamma, \omega-\omega_0] \tag{11}$$

with the abbreviation

$$L[\gamma, \omega] = \frac{1}{\pi}\frac{\gamma}{\gamma^2+\omega^2} \tag{12}$$

for a Lorentz function of area unity. The linewidth (full-width at half-maximum) of the passive resonator ($r_G = 0$) is

$$(\Delta\omega)_{\mathrm{PR}} = \frac{\omega_0}{Q} \tag{13}$$

and the total power P_L (in watts) delivered to the load is given by

$$P_L = \frac{1}{2\pi}\int_0^\infty d\omega S_{v_L, v_L}/R_L. \tag{14}$$

From (11) we find

$$P_L = \frac{S_{v_N, v_N}}{4R_L}\frac{(\Delta\omega)_{\mathrm{PR}}^2}{2\gamma} \tag{15}$$

or, after rearranging,

$$2\gamma = \frac{S_{v_N, v_N}}{4R_L}\frac{(\Delta\omega)_{\mathrm{PR}}^2}{P_L} = kT\frac{R_N}{R_L}\frac{(\Delta\omega)_{\mathrm{PR}}^2}{P_L}. \tag{16}$$

There is a complete analogy between (16) and the expression for the semiconductor laser linewidth (see, e.g., (47) in [5]). The characteristic energy kT in (16) corresponds to the photon energy, P_L corresponds to the laser output power, and $R_N/R_L \approx R_N/R_G$ corresponds to the inversion factor representing the ratio between spontaneous emission rate (the origin of noise) and the optical gain rate. The analogy between electrical quantities used in this model and optical quantities used in laser theory is illustrated in Table I (see also next section).

The normalized power spectral density can be expressed by the linewidth as

$$\frac{S_{v_L, v_L}}{P_L R_L} = \frac{2\gamma}{\gamma^2 + (\omega-\omega_0)^2} \approx \frac{2\gamma}{(\omega-\omega_0)^2} \tag{17}$$

TABLE I
ANALOGY BETWEEN ELECTRICAL AND OPTICAL QUANTITIES

Electrical Quantity	Optical Quantity
Voltage V	Vector potential A
Power spectral density S_{v_L, v_L}	Power spectral density outside the laser
Lorentzian oscillator linewidth 2γ	Homogeneous laser linewidth $(\Delta\omega)_{\text{hom}}$
Oscillation frequency ω_0	Laser frequency ω_{L}
Linewidth of the passive resonator	Linewidth of the passive resonator
Power P_{L} delivered to the load	Laser output power P_{out}
Thermal energy kT	Photon energy $\hbar\omega_{\text{L}}$
Noise-gain ratio $R_{\text{N}}/R_{\text{G}}$	Noise-gain ratio (inversion factor)
Probability density of ω_0	Probability density of ω_{L}
Voltage fluctuation $\Delta V_{\text{C}}(t)$	Velocity $v(t)$ of a diffusing lasing molecule
Probability density of ΔV_{C}	Velocity distribution of the molecules

where, on the right-hand side, a frequency offset $\omega - \omega_0 \gg \gamma$ was assumed. Therefore, the linewidth γ, although hard to measure directly, represents a good measure for the relative phase noise of the oscillator. As for lasers, the Lorentzian linewidth given by (16) is inversely proportional to the power, reflecting the tradeoff between relative phase noise and power consumption.

The question arises how these results can be generalized when a more sophisticated expression of the feedback impedance Z_L is used. Equation (9) may be reexpressed by defining the noise shaping function $|H|^2$ by

$$S_{v_L,\, v_L} = S_{v_N,\, v_N} |H(\omega)|^2 \tag{18}$$

where $H(\omega)$ is the transfer function of the system. Should $1/|H|^2$ not be parabolically shaped, it can be expanded to second order with respect to ω around the oscillation frequency ω_0. Such an expansion is justified if the power spectral density of the freely running oscillator drops with approximately -6 dB per octave, which is often the case. This expansion results in a more general definition of the quality factor

$$Q = \frac{\omega_0}{2} \sqrt{\frac{1}{2} \left. \frac{d^2(1/|H|^2)}{d\omega^2} \right|_{\omega=\omega_0}}. \tag{19}$$

In this case, (9)–(17) remain approximately valid.

IV. The Fluctuating Control Voltage Case

Now we consider the more general case of a fluctuating control voltage $V_C(t) = \langle V_C \rangle + \Delta V_C(t)$, with $\langle V_C \rangle$ denoting the expectation value and $\Delta V_C(t)$ the fluctuations around this value. These fluctuations give rise to fluctuations of the oscillation frequency $\omega_0(t) = \langle \omega_0 \rangle + \Delta\omega_0(t)$. Let the steady-state probability density of $\Delta\omega_0(t)$ be $p(\Delta\omega_0)$. Weighting the result of the previous section with $p(\Delta\omega_0)$, we obtain

$$S_{v_L,\, v_L} = \frac{S_{v_N,\, v_N}}{4} \frac{(\Delta\omega)^2_{\text{PR}}}{\gamma} \pi \cdot \int_{-\infty}^{\infty} d(\Delta\omega_0) p(\Delta\omega_0) L[\gamma,\, \omega - \langle\omega_0\rangle - \Delta\omega_0]. \tag{20}$$

Equation (20) represents a convolution of the Lorentzian with the probability density $p(\Delta\omega_0)$, describing additional line broadening due to low-frequency fluctuations of the control voltage. To obtain a reasonable probability density $p(\Delta\omega_0)$, some assumptions about the stochastic process $\Delta V_C(t)$ or $\Delta\omega_0(t)$ need to made. In the following, we will investigate the influence of phase diffusion on the power spectrum by modeling the time-dependence of the oscillation frequency stochastically. We start with the Langevin equation

$$\frac{d}{dt} \Delta\omega_0(t) + \gamma_{\omega_0} \Delta\omega_0(t) = F(t) \tag{21}$$

where γ_{ω_0} is the damping of the process which drives $\Delta\omega_0(t)$ exponentially toward zero, and $F(t)$ is a white noise force with the autocorrelation function

$$\langle F(t_2) F(t_1) \rangle = 2D\delta(t_2 - t_1). \tag{22}$$

Equations (22) and (23) describe an Ornstein–Uhlenbeck process [6] well known in connection with the motion of a Brownian particle. Its steady-state solution is unique in that it is the only stationary stochastic process that is both Gaussian and Markovian. The $1/f$ noise in semiconductors is not a Gaussian process and, therefore, difficult to model. However, the investigation of the Ornstein–Uhlenbeck process is useful as an approximation, since it gives insight into the interplay between the high-frequency noise in the feedback loop and the low-frequency noise of V_C resulting in total oscillator line broadening. The probability density of the Ornstein–Uhlenbeck process is a Maxwell distribution. This model corresponds to the inhomogeneous line broadening in gas lasers due to the Doppler effect, since the velocity distribution of the lasing molecules is also Maxwellian (see also Table I). The steady-state probability density of $\Delta\omega_0(t)$ is given by

$$p(\Delta\omega_0) = G(\sigma_{\omega_0},\, \Delta\omega_0) \tag{23}$$

with the abbreviation for a Gaussian distribution of unity area

$$G(\sigma_{\omega_0},\, \Delta\omega_0) = \frac{1}{\sqrt{2\pi}\sigma_{\omega_0}} \exp\left[-\frac{(\Delta\omega_0)^2}{2\sigma^2_{\omega_0}}\right] \tag{24}$$

with the rms value given by

$$\sigma_{\omega_0} = \sqrt{\frac{D}{\gamma_{\omega_0}}}. \tag{25}$$

Now we will replace σ_{ω_0} by the rms value σ_{V_C} of the control voltage fluctuations. For this purpose, we assume a linear relationship between the fluctuation $\Delta\omega_0(t)$ and the fluctuation $\Delta V_C(t)$ according to

$$\Delta\omega_0(t) = K\Delta V_C(t) \tag{26}$$

with the sensitivity K defined by $K = d\langle\omega_0\rangle/d\langle V_C\rangle$. Equation (26) implies that the frequency fluctuations instantaneously follow the voltage fluctuations, that is, that the delay time in the control loop is small compared to the characteristic time of the low-frequency fluctuations of $V_C(t)$. The corresponding root-mean-square values are then connected with each other by

$$\sigma_{\omega_0} = |K|\sigma_{V_C}. \tag{27}$$

The root-mean-square value σ_{ω_0} is a measure for the additional line broadening due to the V_C fluctuations. It is a product of the sensitivity $|K|$ and the square root of the integrated noise spectral density of ΔV_C

$$\sigma_{V_C} = \left(\frac{1}{2\pi}\int_0^\infty d\omega S_{\Delta V_C,\,\Delta V_C}(\omega)\right)^{1/2}. \tag{28}$$

Replacing the variance of $\omega_0(t)$ by that of $V_C(t)$ according to (27), we obtain from (23)

$$p(\Delta\omega_0) = G(K\sigma_{V_C},\, \Delta\omega_0). \tag{29}$$

Obviously, there is a trade off between tunability (high $|K|$) and low phase noise (low $|K|$). The Gaussian in (29) needs to be convolved with the Lorentzian according to (20). This gives the final result

$$S_{v_L,\,v_L} = \frac{S_{v_N,\,v_N}}{4}\,\frac{(\Delta\omega)^2_{\mathrm{PR}}}{\gamma}\,\pi V[K\sigma_{V_C},\,\gamma,\,\omega - \langle\omega_0\rangle] \tag{30}$$

with the abbreviation

$$V(\sigma,\,\gamma,\,\omega) = \int_{-\infty}^{\infty} d\omega' G[\sigma,\,\omega']L[\gamma,\,\omega-\omega']. \tag{31}$$

The convolution integral V is called a Voigt line profile and is of unity area. We remark that the derivation of (30) did not require the assumption of an Ornstein–Uhlenbeck process for $\Delta\omega_0(t)$. The only required assumption is that the probability density of the low-frequency fluctuations $\Delta\omega_0(t)$ is Gaussian, which may be a good approximation, even for $1/f$-noise.

Unfortunately, the integral in (31) cannot be performed analytically. A convenient method to calculate the integral is to transform it analytically into the time domain and numerically back into the frequency domain. We obtain then a formula for synthesizing the Voigt line profile

$$V(\sigma,\,\gamma,\,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\exp[j\omega t]\,\exp\left[-\frac{\sigma^2}{2}t^2\right]\exp[-\gamma|t|]. \tag{32}$$

To analyze experimental spectra, several approximations of the Voigt function and its partial derivatives with respect to σ and $\Delta\omega$ are available [7], because the Voigt function plays an important role in optical spectroscopy. Using a least squares fit algorithm for σ and $\Delta\omega$, experimental data can be fitted. A curve fitting procedure based on evolution strategies is described in [8].

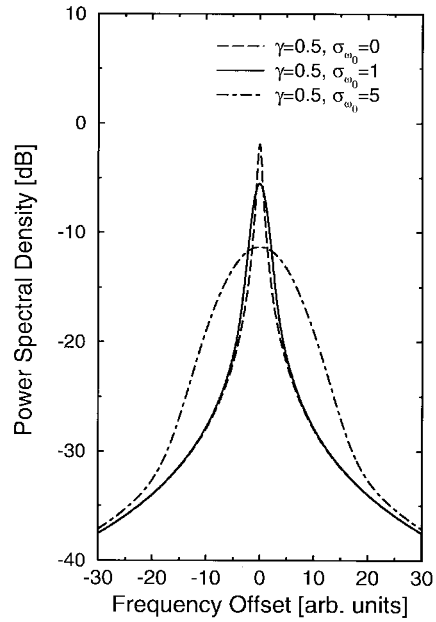

Fig. 2. Normalized power spectral density for the case of vanishing V_C fluctuations (dashed), strong V_C fluctuations (dot-dashed), and medium V_C fluctuations (solid).

Now we will discuss two limiting cases. First, we consider the case of strong V_C fluctuations defined by the relation

$$\sigma_{\omega_0} \gg \gamma \tag{33}$$

with γ the Lorentzian linewidth given by (16). In this case, the Lorentzian L in (31) may be replaced with a δ-function and we find

$$S_{v_L,\,v_L} = \frac{S_{v_N,\,v_N}}{4}\,\frac{(\Delta\omega)^2_{\mathrm{PR}}}{\gamma}\,\pi G(K\sigma_{V_C},\,\omega - \langle\omega_0\rangle). \tag{34}$$

We conclude from (34) that, as for Doppler broadened laser lines, the low-frequency fluctuations are transformed to the vicinity of the oscillation frequency. We point out that the resulting lineshape is determined by the Gaussian *probability* density of these fluctuations and not by the power spectral density which is a Lorentzian for the Ornstein–Uhlenbeck process.

Second, we discuss the opposite limit of weak V_C-fluctuations defined by

$$\sigma_{\omega_0} \ll \gamma. \tag{35}$$

The Gaussian G in (31) may then be replaced with a δ-function, and we find

$$S_{v_L,\,v_L} = \frac{S_{v_N,\,v_N}}{4}\,\frac{(\Delta\omega)^2_{\mathrm{PR}}}{\gamma}\,\pi L[\gamma,\,\omega - \langle\omega_0\rangle] \tag{36}$$

as obtained in the previous section. Figs. 2 and 3 show the lineshapes for these two limiting cases and for a case of medium V_C fluctuations, where σ_{ω_0} and γ are of the same order. In the last case (solid curve), the lineshape resembles a Gaussian for small frequency offsets $\omega - \langle\omega_0\rangle$, and a Lorentzian for large offsets. In other words, if in a semi-logarithmic plot as in Fig. 2 the spectrum is parabolically shaped near the maximum, low-frequency noise contribution of the V_C

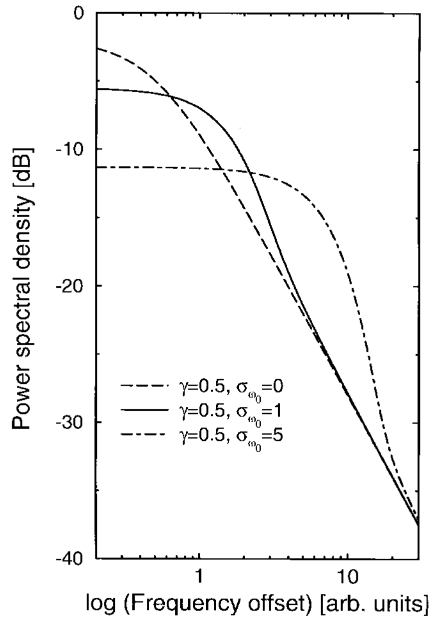

Fig. 3. As Fig. 2, with logarithmic frequency axis.

fluctuations are significant. Such a behavior is observed in the VCO spectrum in [1]. For frequency offsets larger than 8 kHz, the spectrum shows a nearly perfect Lorentzian shape, but at low offsets it is much broader. However, an accurate comparison to our model would require the Voigt function to be convolved with the bandpass filter function $F(\omega)$ of the spectrum analyzer. A corresponding fit to a measured power spectrum would allow the power spectral density to be decomposed into a Lorentzian and a Gaussian contribution, which is, however, beyond the scope of this paper.

V. Application of the Concept to Circuit Design

Up until now, we have considered a very simple oscillator model, since it allows analytical treatment of the phase noise. A detailed investigation of realistic oscillators is beyond the scope of this paper. However, we will explain in the following how the concept may be applied to more sophisticated circuits by combining simulation and measurement with the equations derived above.

Let us assume that the oscillator has been modeled for the case of fixed control voltage. Here the term "control voltage" means either an actual control voltage or a fluctuating supply or substrate voltage. In the absence of flicker noise in the feedback loop, the lineshape will resemble a Lorentzian given by (36). In consideration of the proper normalization, the phase noise with respect to the carrier, also called single-sideband phase noise, is given by

$$S(\omega - \langle\omega_0\rangle) = \frac{2D_\phi}{(\omega - \langle\omega_0\rangle)^2 + D_\phi^2} \tag{37}$$

where the phase diffusivity D_ϕ depends on the circuit.

The probability density function of the control voltage is assumed to be a Gaussian and is, therefore, completely described by the rms value σ_{V_C}. If the sensitivity K is known from simulation or measurement, the rms value of the oscillation frequency can be calculated from (27). The corresponding probability density (24) needs to be convolved with the phase noise spectrum given by (37) as it was done in (31) to obtain the phase noise in consideration of control voltage fluctuations.

VI. Conclusions

We have presented a simple analytical model for the output power spectrum of a voltage-controlled oscillator. The line broadening is due to both the high-frequency noise in the feedback loop and the low-frequency noise of the frequency control voltage. We have stressed the analogy between optical oscillators (lasers) and electrical oscillators allowing one to take advantage of the knowledge in the field of optical spectroscopy. The Lorentzian part of the power spectral density has been expressed in terms of circuit parameters and the consumed power. There is also an additional line broadening due to the noise of the frequency control voltage, which affects the power spectral density, especially for not-too-large frequency offsets. Although we have confined ourselves to a simple oscillator, the methods derived here may be applied to analytical phase noise calculations for more sophisticated circuits including supply and substrate noise.

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