A Simple Model of Feedback Oscillator Noise Spectrum

Introduction

This letter contains brief thoughts on the following points.

1) The relationships among four commonly used spectral descriptions of oscillator short-term stability or noise behavior.
2) A heuristic derivation, presented without formal proof, of the expected spectrum of a feedback oscillator in terms of known oscillator parameters.
3) Some experimental results which illustrate the validity of the simple model.
4) Comments on the effect of nonlinearity, specific spectral requirements for several applications, choice of resonator frequency and active element, and expected spectrum characteristics of several oscillator types.

Spectral Models of Phase Variations

Consider a stable oscillator whose measurable output can be expressed as

$$v(t) = A \cos \left[\omega_0 t + \phi(t)\right].$$

It is common to treat $\phi(t)$ as a zero-mean stationary random process describing deviations of the phase from the ideal. The frequency domain information about phase or frequency variations is contained in the "power" spectral density $S_{\dot{\phi}}(\omega_m)$ of the phase $\phi(t)$ or, alternatively, in the "power" spectral density $S_{\phi}(\omega_m)$ of the frequency $\dot{\phi}$. By analogy to modulation theory, we use ω_m to mean the modulation, video, baseband, or offset frequency associated with the noise-like variations in $\phi(t)$. The units of $S_{\phi}(\omega_m)$ are radians^2/cps bandwidth or dB relative to 1 radians^2/cps BW; $S_{\dot{\phi}}(\omega_m)$ is expressed in (radians/sec)^2 per c/s BW [1] [2]. The two are related by $S_{\dot{\phi}}(\omega_m) = \omega_m{}^2 S_{\phi}(\omega_m)$.

$S_{\dot{\phi}}(\omega_m)$ can also be expressed in terms of the equivalent rms frequency deviation Δf_{rms} in a given video bandwidth. Further, subject to the limitations that $\overline{\phi^2} \ll 1$ (small total modulation indes) and that AM$\ll$FM components, the normalized RF power spectrum $G(\omega-\omega_0)$ is identical to the two-sided spectrum of the phase $S_{\phi}(\omega_m)$; i.e., RF sidebands relative to the carrier are down by $S_{\phi}(\omega_m)$ expressed in decibels relative to 1 radian^2/BW.

Relation to Oscillator Internal Noise

A basic requirement on an oscillator noise model is that it show clearly the relationship of the spectrum of the phase $S_{\phi}(\omega_m)$ to the known or expected noise and signal levels and resonator characteristics of the oscillator. A simple picture can be constructed using a model of a linear feedback oscillator. Minor corrections to the results are necessary to account for nonlinear effects which must be present in a physical oscillator. Assume a single resonator feedback network of fractional bandwidth $2B/\omega_0 = 1/Q$, where Q is the operating, or loaded, quality factor. For small phase deviations at video rates which fall within the feedback half-bandwidth $\omega_0/2Q$, a phase error at the oscillator input due to noise or parameter variations results in a frequency error determined by the phase-frequency relationship of the feedback network, $\Delta\theta = 2Q\dot{\phi}/\omega_0$. Thus, for modulation rates *less* than the half-bandwidth of the feedback loop, the spectrum of the *frequency* $S_{\dot{\phi}}(\omega)$ is identical (with a scale factor) to the spectrum of the uncertainty of the oscillator input phase due to noise and parameter variations. This uncertainty will be denoted $\Delta\theta(t)$, and its two-sided power spectral density $S_{\Delta\theta}(\omega_m)$.

For modulation rates *large* compared to the feedback bandwidth, a series feedback loop is out of the circuit. At these modulation rates, the power spectral density of the output *phase* $S_{\phi}(\omega)$ is identical to the spectrum of the oscillator input phase uncertainty $S_{\Delta\theta}(\omega_m)$.

For a physical oscillator the spectrum $S_{\Delta\theta}(\omega_m)$ of the input phase uncertainty $\Delta\theta(t)$ is expected to have two principal components. One component is due to phase uncertainties resulting from additive white noise at frequencies around the oscillator frequency, as well as noise at other frequencies mixed into the pass band of interest by nonlinearities. The second component is due to parameter variations at video frequencies which affect the phase (such as variations in the phase shift of a transistor due to carrier density fluctuations in the base resistance). The additive noise component of $S_{\Delta\theta}(\omega_m)$ is identical to the spectral density of the noise voltage squared relative to the mean square signal voltage. For white additive noise, this component is flat with frequency. For a feedback oscillator with an effective noise figure F, the two-sided $S_{\Delta\theta}(\omega) = 2\mathrm{FKT}/P_S$; P_S is the signal level at oscillator active element input.

The video spectrum of parameter variations is found typically to have a power spectral density varying inversely with frequency (a $1/\omega_m$ or $1/f$ spectrum). The total power spectral density of oscillator input phase errors is of the form $S_{\Delta\theta}(\omega_m) = \alpha/\omega_m + \beta$, where α is a constant determined by the level of $1/f$ variations and β is $= 2\mathrm{FKT}/P_S$ for two-sided spectra.

To find $S_{\phi}(\omega_m)$ or $S_{\dot{\phi}}(\omega_m)$, we use the fact that

$$\text{for} \quad \omega_m < \frac{\omega_0}{2Q} \qquad S_{\phi}(\omega) = \left[\frac{\omega_0}{2Q}\right]^2 S_{\Delta\theta}(\omega_m)$$

$$\omega_m > \frac{\omega_0}{2Q} \qquad S_{\dot{\phi}}(\omega) = S_{\Delta\theta}(\omega_m).$$

A suitable composite expression is

$$S_{\phi}(\omega_m) = S_{\Delta\theta}\left[1 + \left(\frac{\omega_0}{2Q\omega_m}\right)^2\right].$$

This yields an asymptotic model for $S_{\phi}(\omega)$ shown on log-log scales in Fig. 1.

The model can be summarized as follows.

$S_{\phi}(\omega_m)$ decreases with ω_m
- at 9 dB/octave up to the point where $1/f$ effects no longer predominate.
- at 6 dB/octave from that point up to the feedback loop half-bandwidth.
- at 0 dB/octave above that frequency up to a limit imposed by subsequent filtering.

$S_{\dot{\phi}}(\omega)$ decreases at 3 dB/octave up to the first breakpoint, is flat with frequency up to the feedback baseband bandwidth, and increases at 6 dB octave above that point.

The case where $1/f$ effects predominate only for frequencies small compared with the feedback loop bandwidth is shown here as an

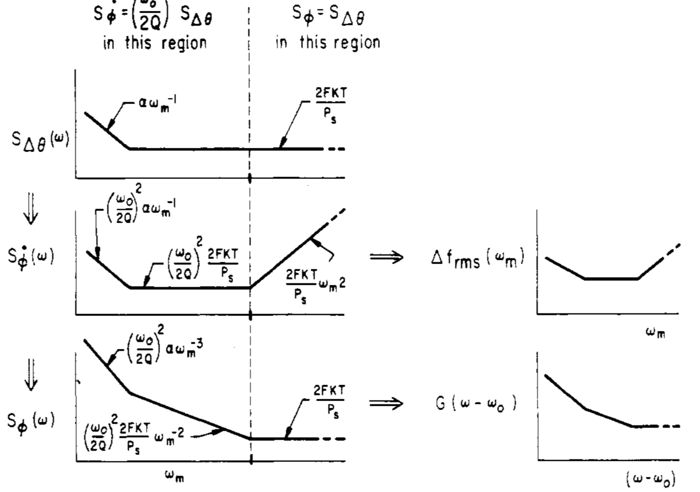

Fig. 1. Derivation of Oscillator Spectra. The logical sequence leading from oscillator parameters to spectrum characteristics is presented here. The power spectra of output phase or frequency are derived from the spectrum of input phase uncertainties and from the oscillator feedback bandwidth. The calculable constants of the oscillator are FKT, P_s, and $\omega_0/2Q$; the $1/f$ constant α is not accurately predictable but can be inferred from data. The amplitude spectrum of frequency deviation and the RF spectrum can be derived as shown, subject to limitations discussed in the text.

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example. For a high-Q oscillator, $1/f$ effects in $S_{\Delta\theta}$ can predominate out to a modulation rate exceeding $\omega_0/2Q$; in this case there is no 6 dB per octave region in $S_\phi(\omega)$. A similar spectrum results where large additive noise in following amplifier stages or measuring equipment obscures the oscillator internal noise, except at very low modulation frequencies.

Note that there is a portion of the curve $S_\phi(\omega_m)$ which is proportional to $1/\omega_m^2$, leading to a $1/\omega_m$ or $1/f$ variation for rms phase deviation. This is often confused with the true $1/f$ effects associated with parameter variations leading to the $1/f$ portion of the curve for $S_{\Delta\theta}(\omega_m)$ and $S_\phi(\omega_m)$. These two are not the same thing; "$1/f$" refers to a power spectral density rather than an amplitude spectrum.

In practice, the measurable $S_\phi(\omega_m)$ is always modified by subsequent bandlimiting filtering and by additive noise contributed by following amplifiers. It is conceivable that, for a two-terminal oscillator, the filtering action of the resonator eliminates the additive phase noise component for $\omega_m > (\omega_0/2Q)$.

Experimental Verification

Measurements were taken on a stable microwave signal source[1] designed to have a spectral purity limited only by the oscillator, which was a 100 M/s crystal oscillator. This unit employs two large-jump step recovery diode multipliers with amplification between them. The data are presented in Fig. 2 in comparison with a model derived from the following constants:

Feedback bandwidth	= 16 kc/s
P_S	= −4 dBm
F	= 9 dB
KT	= −174 dBm in 1 c/s BW
Multiplication ratio	= 100 = 40 dB
$N^2 2FKT/P_S$	= +40+3+9−174+4 = −118 dB.

This leads to an asymptotic value for $S_\phi(\omega_m)$ of −118 dB relative to 1 radian²/BW in 1 c/s bandwidth, i.e., a carrier-to-sideband ratio of 118 dB. The "$1/f$" region (9 dB/octave) constant α is estimated for best data fit.

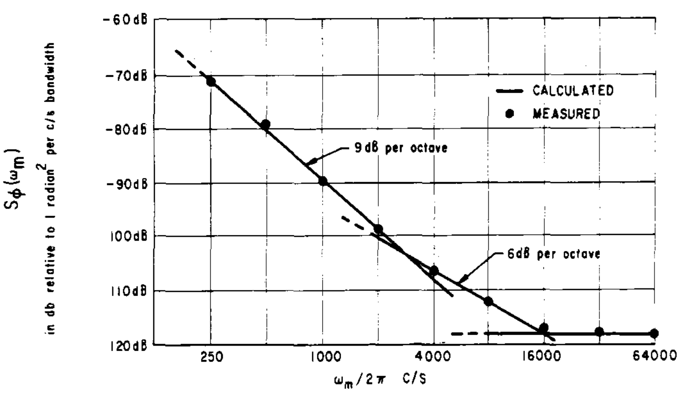

Fig. 2. $S_\phi(\omega_m)$ for Stable Microwave Signal Source. The data presented here is the average of two independent measurements which were in excellent agreement. These measurements were made at X Band on the multiplied output of a 100 Mc/s voltage controlled crystal oscillator having a 16 kc/s feedback half-bandwidth. Since this bandwidth can be reduced by a considerable factor without exceeding the present state of the art, the data is not intended to represent ultimate attainable levels, but rather serves as an illustrative example. The $1/f$ constant is chosen for best data fit. Slopes and other calculated parameters are derived from known oscillator characteristics.

Nonlinear Effects

The data was based on an estimated transistor noise figure of 9 dB. This was taken high to account for nonlinear mixing of noise at third harmonic and higher frequencies which is mixed into the pass band of interest by second harmonic periodic parameter variations caused by the nonlinearity. The excellent fit of the data implies that this degradation of effective noise figure may well be an adequate description of the effect of nonlinearity.

Video Frequency Range of Interest

A number of applications which have been dealt with in this issue of the Proceedings may be summarized in terms of the video frequency range of interest. Space systems and Doppler radar applications are of particular interest to the author. For these two, interest lies in the range of a few c/s up to 100 kc/s. Space applications typically concentrate on the range where, for a crystal oscillator, $S_\phi(\omega_m)$ is proportional to $S_{\Delta\theta}(\omega_m)$ [3], while Doppler radar applications place additional emphasis on the region above the oscillator feedback loop bandwidth [4]. Both applications typically require microwave systems which employ multiplication from the oscillator frequency.

Choice of Oscillator Frequency for Crystal Oscillator-Multiplier

It is of interest to inspect the effect of oscillator frequency upon the output spectrum of an oscillator-multiplier system having a fixed output frequency. Two assumptions which aid the calculation are a) constant oscillator input signal-to-noise ratio, and b) resonator Q varying inversely with the oscillator frequency ω_0. Under these assumptions a comparison of two oscillator frequencies yields the following results.

1) For $\omega_m < (\omega_0/2Q)$, of the lower frequency oscillator, the multiplied output $S_\phi(\omega_m)$ is identical for either choice.
2) For $\omega_m \gg (\omega_0/2Q)$, the output $S_\phi(\omega_m)$ varies as the square of the multiplication ratio (i.e., inversely as the square of the oscillator frequency).

This can be verified by a simple graphical construction.

Choice of Active Element in a Transistor Oscillator

It is apparent that $1/f$ variations and nonlinearity can have significant deleterious effects on the attainable low levels of $S_\phi(\omega_m)$. In the light of suggestions by O. Mueller that microthermal effects [6] contribute to $1/f$ noise in transistors, it is suggested that AGC oscillators using large area transistors having high power capabilities may provide simultaneous improvements in $1/f$ level and in nonlinear effects.

Spectrum Characteristics of Microwave Solid State Sources

The spectrum model given here allows simple prediction of spectrum shape and level for microwave sources of the types discussed by Johnson et al [5]. Comparison with their data shows good agreement —their measurements for crystal oscillator units extend to $\omega_m \gg (\omega_0/2Q)$, while microwave oscillators are characterized by Q factors such that, for the measurements cited, $\omega_m < (\omega_0/2Q)$.

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[1] 9.5 Gc/s Solid State Local Oscillator PN 31-007191, manufactured by Applied Technology, Inc., Palo Alto, Calif. Measurements are average of values measured by the author and D. J. Healey, III, Westinghouse Corp., Baltimore, Md., using Spectra Electronics SE-200 and Westinghouse proprietary noise test sets.