

# Ampère's circuital law

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In classical electromagnetism, **Ampère's circuital law**, discovered by André-Marie Ampère in 1826,<sup>[1]</sup> relates the integrated magnetic field around a closed loop to the electric current passing through the loop. James Clerk Maxwell derived it again using hydrodynamics in his 1861 paper *On Physical Lines of Force* and it is now one of the Maxwell equations, which form the basis of classical electromagnetism.

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## Ampère's original circuital law

Ampère's law relates magnetic fields to electric currents that produce them. Ampère's law determines the magnetic field associated with a given current, or the current associated with a given magnetic field, provided that the electric field does not change over time. In its original form, Ampère's circuital law relates a magnetic field to its electric current source. The law can be written in two forms, the "integral form" and the "differential form". The forms are equivalent, and related by the Kelvin–Stokes theorem. It can also be written in terms of either the **B** or **H** magnetic fields. Again, the two forms are equivalent (see the "proof" section below).

Ampère's circuital law is now known to be a correct law of physics in a magnetostatic situation: The system is static except possibly for continuous steady currents within closed loops. In all other cases the law is incorrect unless Maxwell's correction is included (see below).

### Integral form

In SI units (cgs units are later), the "integral form" of the original Ampère's circuital law is a line integral of the magnetic field around some closed curve *C* (arbitrary but must be closed). The curve *C* in turn bounds both a surface *S* which the electric current passes through (again arbitrary but not closed—since no three-dimensional volume is enclosed by *S*), and encloses the current. The mathematical statement of the law is a relation between the total amount of magnetic field around some path (line integral) due to the current which passes through that enclosed path (surface integral). It can be written in a number of forms.<sup>[2][3]</sup>

In terms of **total** current, which includes both **free** and **bound** current, the line integral of the magnetic B-field (in tesla, T) around closed curve *C* is proportional to the total current *I*<sub>enc</sub> passing through a surface *S* (enclosed by *C*):

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S} = \mu_0 I_{\text{enc}}$$

where **J** is the total current density (in ampere per square metre, Am<sup>−2</sup>).

Alternatively in terms of **free** current, the line integral of the magnetic H-field (in ampere per metre, Am<sup>−1</sup>) around closed curve *C* equals the free current *I*<sub>f,enc</sub> through a surface *S*:

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \iint_S \mathbf{J}_f \cdot d\mathbf{S} = I_{f,\text{enc}}$$

where **J**<sub>f</sub> is the free current density only. Furthermore

- $\oint_C$  is the closed line integral around the closed curve  $C$ ,
- $\iint_S$  denotes a 2d surface integral over  $S$  enclosed by  $C$
- $\cdot$  is the vector dot product,
- $d\mathbf{l}$  is an infinitesimal element (a differential) of the curve  $C$  (i.e. a vector with magnitude equal to the length of the infinitesimal line element, and direction given by the tangent to the curve  $C$ )
- $d\mathbf{S}$  is the vector area of an infinitesimal element of surface  $S$  (that is, a vector with magnitude equal to the area of the infinitesimal surface element, and direction normal to surface  $S$ . The direction of the normal must correspond with the orientation of  $C$  by the right hand rule), see below for further explanation of the curve  $C$  and surface  $S$ .

The  $\mathbf{B}$  and  $\mathbf{H}$  fields are related by the constitutive equation

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where  $\mu_0$  is the magnetic constant.

There are a number of ambiguities in the above definitions that require clarification and a choice of convention.

1. First, three of these terms are associated with sign ambiguities: the line integral  $\oint_C$  could go around the loop in either direction (clockwise or counterclockwise); the vector area  $d\mathbf{S}$  could point in either of the two directions normal to the surface; and  $I_{\text{enc}}$  is the net current passing through the surface  $S$ , meaning the current passing through in one direction, minus the current in the other direction—but either direction could be chosen as positive. These ambiguities are resolved by the right-hand rule: With the palm of the right-hand toward the area of integration, and the index-finger pointing along the direction of line-integration, the outstretched thumb points in the direction that must be chosen for the vector area  $d\mathbf{S}$ . Also the current passing in the same direction as  $d\mathbf{S}$  must be counted as positive. The right hand grip rule can also be used to determine the signs.
2. Second, there are infinitely many possible surfaces  $S$  that have the curve  $C$  as their border. (Imagine a soap film on a wire loop, which can be deformed by moving the wire). Which of those surfaces is to be chosen? If the loop does not lie in a single plane, for example, there is no one obvious choice. The answer is that it does not matter; it can be proven that any surface with boundary  $C$  can be chosen.

## Differential form

By the Stokes' theorem, this equation can also be written in a "differential form". Again, this equation only applies in the case where the electric field is constant in time, meaning the currents are steady (time-independent, else the magnetic field would change with time); see below for the more general form. In SI units, the equation states for total current:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

and for free current

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

where  $\nabla \times$  is the curl operator.

## Note on free current versus bound current

The electric current that arises in the simplest textbook situations would be classified as "free current"—for example, the current that passes through a wire or battery. In contrast, "bound current" arises in the context of bulk materials that can be magnetized and/or polarized. (All materials can to some extent.)

When a material is magnetized (for example, by placing it in an external magnetic field), the electrons remain bound to their respective atoms, but behave as if they were orbiting the nucleus in a particular direction, creating a microscopic current. When the currents from all these atoms are put together, they create the same effect as a macroscopic current, circulating perpetually around the magnetized object. This magnetization current  $\mathbf{J}_M$  is one contribution to "bound current".

The other source of bound current is bound charge. When an electric field is applied, the positive and negative bound charges can separate over atomic distances in polarizable materials, and when the bound charges move, the polarization changes, creating another contribution to the "bound current", the polarization current  $\mathbf{J}_P$ .

The total current density  $\mathbf{J}$  due to free and bound charges is then:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_M + \mathbf{J}_P$$

with **J**<sub>f</sub> the "free" or "conduction" current density.

All current is fundamentally the same, microscopically. Nevertheless, there are often practical reasons for wanting to treat bound current differently from free current. For example, the bound current usually originates over atomic dimensions, and one may wish to take advantage of a simpler theory intended for larger dimensions. The result is that the more microscopic Ampère's law, expressed in terms of **B** and the microscopic current (which includes free, magnetization and polarization currents), is sometimes put into the equivalent form below in terms of **H** and the free current only. For a detailed definition of free current and bound current, and the proof that the two formulations are equivalent, see the "proof" section below.

### Shortcomings of the original formulation of Ampère's circuital law

There are two important issues regarding Ampère's law that require closer scrutiny. First, there is an issue regarding the continuity equation for electrical charge. In vector calculus, the identity for the divergence of a curl states that a vector field's curl divergence must always be zero. Hence

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

and so the original Ampère's law implies that

$$\nabla \cdot \mathbf{J} = 0.$$

But in general

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

which is non-zero for a time-varying charge density. An example occurs in a capacitor circuit where time-varying charge densities exist on the plates.<sup>[4][5][6][7][8]</sup>

Second, there is an issue regarding the propagation of electromagnetic waves. For example, in free space, where

$$\mathbf{J} = \mathbf{0}.$$

Ampère's law implies that

$$\nabla \times \mathbf{B} = \mathbf{0}$$

but instead

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$

To treat these situations, the contribution of displacement current must be added to the current term in Ampère's law.

James Clerk Maxwell conceived of displacement current as a polarization current in the dielectric vortex sea, which he used to model the magnetic field hydrodynamically and mechanically.<sup>[9]</sup> He added this displacement current to Ampère's circuital law at equation (112) in his 1861 paper *On Physical Lines of Force*.<sup>[10]</sup>

### Displacement current

In free space, the displacement current is related to the time rate of change of electric field.

In a dielectric the above contribution to displacement current is present too, but a major contribution to the displacement current is related to the polarization of the individual molecules of the dielectric material. Even though charges cannot flow freely in a dielectric, the charges in molecules can move a little under the influence of an electric field. The positive and negative charges in molecules separate under the applied field, causing an increase in the state of polarization, expressed as the polarization density **P**. A changing state of polarization is equivalent to a current.

Both contributions to the displacement current are combined by defining the displacement current as:<sup>[4]</sup>

$$\mathbf{J}_D = \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t),$$

where the electric displacement field is defined as:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E},$$

where  $\epsilon_0$  is the electric constant,  $\epsilon_r$  the relative static permittivity, and **P** is the polarization density. Substituting this form for **D** in the expression for displacement current, it has two components:

$$\mathbf{J}_D = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}.$$

The first term on the right hand side is present everywhere, even in a vacuum. It doesn't involve any actual movement of charge, but it nevertheless has an associated magnetic field, as if it were an actual current. Some authors apply the name *displacement current* to only this contribution.<sup>[11]</sup>

The second term on the right hand side is the displacement current as originally conceived by Maxwell, associated with the polarization of the individual molecules of the dielectric material.

Maxwell's original explanation for displacement current focused upon the situation that occurs in dielectric media. In the modern post-aether era, the concept has been extended to apply to situations with no material media present, for example, to the vacuum between the plates of a charging vacuum capacitor. The displacement current is justified today because it serves several requirements of an electromagnetic theory: correct prediction of magnetic fields in regions where no free current flows; prediction of wave propagation of electromagnetic fields; and conservation of electric charge in cases where charge density is time-varying. For greater discussion see Displacement current.

## Extending the original law: the Maxwell–Ampère equation

Next Ampère's equation is extended by including the polarization current, thereby remedying the limited applicability of the original Ampère's circuital law.

Treating free charges separately from bound charges, Ampère's equation including Maxwell's correction in terms of the **H**-field is (the **H**-field is used because it includes the magnetization currents, so **J**<sub>M</sub> does not appear explicitly, see H-field and also Note):<sup>[12]</sup>

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \iint_S \left( \mathbf{J}_f + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot d\mathbf{S}$$

(integral form), where **H** is the magnetic H field (also called "auxiliary magnetic field", "magnetic field intensity", or just "magnetic field"), **D** is the electric displacement field, and **J**<sub>f</sub> is the enclosed conduction current or free current density. In differential form,

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial}{\partial t} \mathbf{D}.$$

On the other hand, treating all charges on the same footing (disregarding whether they are bound or free charges), the generalized Ampère's equation, also called the Maxwell–Ampère equation, is in integral form (see the "proof" section below):

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \iint_S \left( \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E} \right) \cdot d\mathbf{S}$$

In differential form,

$$\nabla \times \mathbf{B} = \left( \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E} \right)$$

In both forms **J** includes magnetization current density<sup>[13]</sup> as well as conduction and polarization current densities. That is, the current density on the right side of the Ampère–Maxwell equation is:

$$\mathbf{J}_f + \mathbf{J}_D + \mathbf{J}_M = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

where current density **J**<sub>D</sub> is the *displacement current*, and **J** is the current density contribution actually due to movement of charges, both free and bound. Because  $\nabla \cdot \mathbf{D} = \rho$ , the charge continuity issue with Ampère's original formulation is no longer a problem.<sup>[14]</sup> Because of the term in  $\epsilon_0 \partial \mathbf{E} / \partial t$ , wave propagation in free space now is possible.

With the addition of the displacement current, Maxwell was able to hypothesize (correctly) that light was a form of electromagnetic wave. See electromagnetic wave equation for a discussion of this important discovery.

### Proof of equivalence

**Proof that the formulations of Ampère's law in terms of free current are equivalent to the formulations involving total current.**

In this proof, we will show that the equation

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

is equivalent to the equation

$$\nabla \times \mathbf{B}/\mu_0 = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Note that we're only dealing with the differential forms, not the integral forms, but that is sufficient since the differential and integral forms are equivalent in each case, by the Kelvin–Stokes theorem.

We introduce the polarization density  $\mathbf{P}$ , which has the following relation to  $\mathbf{E}$  and  $\mathbf{D}$ :

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Next, we introduce the magnetization density  $\mathbf{M}$ , which has the following relation to  $\mathbf{B}$  and  $\mathbf{H}$ :

$$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$$

and the following relation to the bound current:

$$\begin{aligned} \mathbf{J}_{\text{bound}} &= \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} , \\ &= \mathbf{J}_M + \mathbf{J}_P , \end{aligned}$$

where

$$\mathbf{J}_M = \nabla \times \mathbf{M} ,$$

is called the magnetization current density, and

$$\mathbf{J}_P = \frac{\partial \mathbf{P}}{\partial t} ,$$

is the polarization current density. Taking the equation for  $\mathbf{B}$ :

$$\begin{aligned} \nabla \times \mathbf{B}/\mu_0 &= \nabla \times (\mathbf{H} + \mathbf{M}) \\ &= \nabla \times \mathbf{H} + \mathbf{J}_M \\ &= \mathbf{J}_f + \mathbf{J}_P + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_M \end{aligned}$$

Consequently, referring to the definition of the bound current:

$$\begin{aligned} \nabla \times \mathbf{B}/\mu_0 &= \mathbf{J}_f + \mathbf{J}_{\text{bound}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ &= \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} , \end{aligned}$$

as was to be shown.

## Ampère's law in cgs units

In cgs units, the integral form of the equation, including Maxwell's correction, reads

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \frac{1}{c} \iint_S \left( 4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S}$$

where  $c$  is the speed of light.

The differential form of the equation (again, including Maxwell's correction) is

$$\nabla \times \mathbf{B} = \frac{1}{c} \left( 4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right) .$$

## See also

- Biot–Savart law
- Displacement current
- Capacitance
- Ampèrian magnetic dipole model
- Electromagnetic wave equation
- Maxwell's equations
- Faraday's law of induction
- Bound charge
- Electric current
- Vector calculus
- Stokes' theorem

## Notes

- ↑ Richard Fitzpatrick (2007). "Ampère's Circuital Law" (<http://farside.ph.utexas.edu/teaching/316/lectures/node75.html>).
- ↑ Heinz E Knoepfel (2000). *Magnetic Fields: A comprehensive theoretical treatise for practical use* (<http://books.google.com/?id=1n7NPasgh0sC&pg=PA210&dq=isbn=0-471-32205-9#PPA4,M1>). Wiley. p. 4. ISBN 0-471-32205-9.
- ↑ George E. Owen (2003). *Electromagnetic Theory* ([http://books.google.com/?id=VLm\\_dqhZUOYC&pg=PA213&dq=%22Ampere%27s+circuital+law%22](http://books.google.com/?id=VLm_dqhZUOYC&pg=PA213&dq=%22Ampere%27s+circuital+law%22)) (Reprint of 1963 ed.). Courier-Dover Publications. p. 213. ISBN 0-486-42830-3.
- ↑ <sup>*a*</sup> <sup>*b*</sup> John David Jackson (1999). *Classical Electrodynamics* (3rd ed.). Wiley. p. 238. ISBN 0-471-30932-X.
- ↑ David J Griffiths (1999). *Introduction to Electrodynamics* (<http://books.google.com/?id=w0YgJgAACAAJ&dq=isbn=0-13-805326-X>) (3rd ed.). Pearson/Addison-Wesley. pp. 322–323. ISBN 0-13-805326-X.
- ↑ George E. Owen (2003). op. cit. ([http://books.google.com/?id=VLm\\_dqhZUOYC&pg=PA213&dq=%22Ampere%27s+circuital+law%22](http://books.google.com/?id=VLm_dqhZUOYC&pg=PA213&dq=%22Ampere%27s+circuital+law%22)). Mineola, N.Y.: Dover Publications. p. 285. ISBN 0-486-42830-3.
- ↑ J. Billingham, A. C. King (2006). *Wave Motion* (<http://books.google.com/?id=bNePaHM20LQC&pg=PA179&dq=displacement+%22ampere%27s+law%22>). Cambridge University Press. p. 179. ISBN 0-521-63450-4.
- ↑ J.C. Slater and N.H. Frank (1969). *Electromagnetism* (<http://books.google.com/?id=GYsphnFwUuUC&pg=PA83&dq=displacement+%22ampere%27s+law%22>) (Reprint of 1947 ed.). Courier Dover Publications. p. 83. ISBN 0-486-62263-0.
- ↑ Daniel M. Siegel (2003). *Innovation in Maxwell's Electromagnetic Theory: Molecular Vortices, Displacement Current, and Light* (<http://books.google.com/?id=AbQq85U8K0gC&pg=PA97&dq=Ampere%27s+circuital+law+%22displacement+current%22>). Cambridge University Press. pp. 96–98. ISBN 0-521-53329-5.
- ↑ James C. Maxwell (1861). "On Physical Lines of Force" ([http://upload.wikimedia.org/wikipedia/commons/b/b8/On\\_Physical\\_Lines\\_of\\_Force.pdf](http://upload.wikimedia.org/wikipedia/commons/b/b8/On_Physical_Lines_of_Force.pdf)). *Philosophical Magazine and Journal of Science*.
- ↑ For example, see David J. Griffiths (1999). op. cit.. Upper Saddle River, NJ: Prentice Hall. p. 323. ISBN 0-13-805326-X. and Tai L. Chow (2006). *Introduction to Electromagnetic Theory* (<http://books.google.com/?id=dpnpMhw1zo8C&pg=PA153&dq=isbn=0-7637-3827-1#PPA204,M1>). Jones & Bartlett. p. 204. ISBN 0-7637-3827-1.
- ↑ Mircea S. Rogalski, Stuart B. Palmer (2006). *Advanced University Physics* (<http://books.google.com/?id=rW95PWh3YbkC&pg=PA266&dq=displacement+%22Ampere%27s+circuital+law%22>). CRC Press. p. 267. ISBN 1-58488-511-4.
- ↑ Stuart B. Palmer & Mircea S. Rogalski (2006). *Advanced University Physics* (<http://books.google.com/?id=rW95PWh3YbkC&printsec=frontcover#PPA251,M1>). CRC Press. p. 251. ISBN 1-58488-511-4.
- ↑ The magnetization current can be expressed as the *curl* of the magnetization, so its divergence is zero and it does not contribute to the continuity equation. See magnetization current.

## Further reading

- Griffiths, David J. (1998). *Introduction to Electrodynamics (3rd ed.)*. Prentice Hall. ISBN 0-13-805326-X.
- Tipler, Paul (2004). *Physics for Scientists and Engineers: Electricity, Magnetism, Light, and Elementary Modern Physics (5th ed.)*. W. H. Freeman. ISBN 0-7167-0810-8.

## External links

- Simple Nature* by Benjamin Crowell ([http://www.lightandmatter.com/html\\_books/0sn/ch11/ch11.html#Section11.3](http://www.lightandmatter.com/html_books/0sn/ch11/ch11.html#Section11.3)) Ampere's law from an online textbook
- MISN-0-138 *Ampere's Law* ([http://www.physnet.org/modules/pdf\\_modules/m138.pdf](http://www.physnet.org/modules/pdf_modules/m138.pdf)) (PDF file) by Kirby Morgan for Project PHYSNET (<http://www.physnet.org>).

- MISN-0-145 *The Ampere–Maxwell Equation; Displacement Current* ([http://www.physnet.org/modules/pdf\\_modules/m145.pdf](http://www.physnet.org/modules/pdf_modules/m145.pdf)) (PDF file) by J.S. Kovacs for Project PHYSNET.
- *The Ampère's Law Song* (<http://www.haverford.edu/physics-astro/songs/ampere.PDF>) (PDF file) by Walter Fox Smith; Main page (<http://www.haverford.edu/physics-astro/songs/>), with recordings of the song.
- *A Dynamical Theory of the Electromagnetic Field* ([http://upload.wikimedia.org/wikipedia/commons/1/19/A\\_Dynamical\\_Theory\\_of\\_the\\_Electromagnetic\\_Field.pdf](http://upload.wikimedia.org/wikipedia/commons/1/19/A_Dynamical_Theory_of_the_Electromagnetic_Field.pdf)) Maxwell's paper of 1864

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