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Stokes parameters

The **Stokes parameters** are a set of values that describe the polarization state of electromagnetic radiation. They were defined by George Gabriel Stokes in 1852,[1] as a mathematically convenient alternative to the more common description of incoherent or partially polarized radiation in terms of its total intensity (I), (fractional) degree of polarization (p), and the shape parameters of the polarization ellipse. The effect of an optical system on the polarization of light can be determined by constructing the Stokes vector for the input light and applying Mueller calculus, to obtain the Stokes vector of the light leaving the system.

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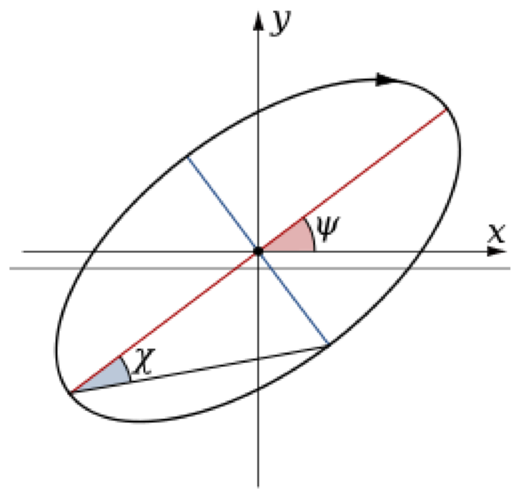
Definitions

The relationship of the Stokes parameters S_0, S_1, S_2, S_3 to intensity and polarization ellipse parameters is shown in the equations below and the figure at right.

$$\begin{aligned}S_0 &= I\\ S_1 &= Ip\cos 2\psi \cos 2\chi\\ S_2 &= Ip\sin 2\psi \cos 2\chi\\ S_3 &= Ip\sin 2\chi\end{aligned}$$

Here I, p, 2ψ and 2χ are the spherical coordinates of the three-dimensional vector of cartesian coordinates

(S_1, S_2, S_3). I is the total intensity of the beam, and p is the degree of polarization, constrained by $0 \le p \le 1$. The factor of two before ψ represents the fact that any polarization ellipse is indistinguishable from one rotated by 180°, while the factor of two before χ indicates that an ellipse is indistinguishable from one with the semi-axis lengths swapped accompanied by a 90° rotation. The phase information of the polarized light is not recorded in the Stokes parameters. The four Stokes parameters are sometimes denoted I, Q, U and V, respectively.



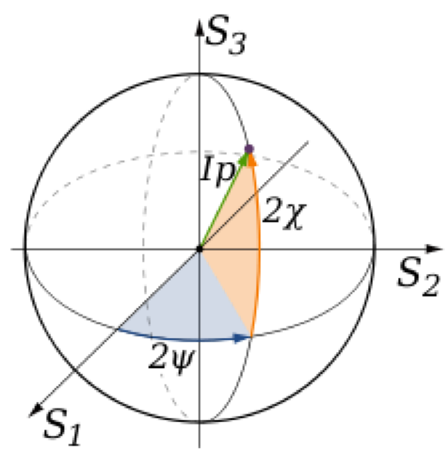
If given the Stokes parameters one can solve for the spherical coordinates with the following equations:

$$I = S_0$$

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

$$2\psi = \mathrm{atan}\frac{S_2}{S_1}$$

$$2\chi = \mathrm{atan}\frac{S_3}{\sqrt{S_1^2 + S_2^2}}$$



The Poincaré sphere is the parametrisation of the last three Stokes' parameters in spherical coordinates

Stokes vectors

The Stokes parameters are often combined into a vector, known as the **Stokes vector**:

$$\vec{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

The Stokes vector spans the space of unpolarized, partially polarized, and fully polarized light. For comparison, the Jones vector only spans the space of fully polarized light, but is more useful for problems involving coherent light. The four Stokes parameters are not a preferred coordinate system of the space, but rather were chosen because they can be easily measured or calculated.

Note that there is an ambiguous sign for the V component depending on the physical convention used. In practice, there are two separate conventions used, either defining the Stokes parameters when looking down the beam towards the source (opposite the direction of light propagation) or looking down the beam away from the source (coincident with the direction of light propagation). These two conventions result in different signs for V, and a convention must be chosen and adhered to.

Examples

Below are shown some Stokes vectors for common states of polarization of light.

$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ Linearly polarized (horizontal)

$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$ Linearly polarized (vertical)

$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ Linearly polarized (+45°)

$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$ Linearly polarized (−45°)

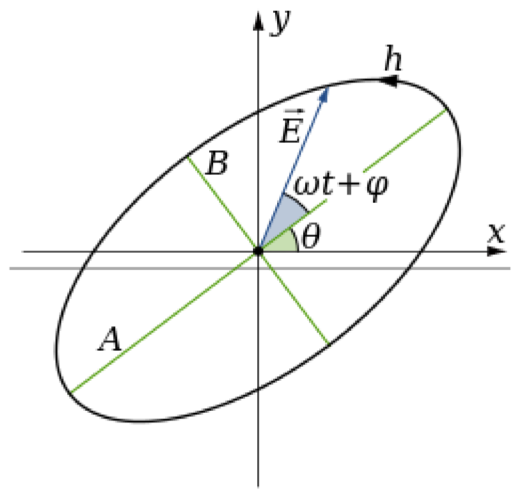
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ Right-hand circularly polarized

$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$ Left-hand circularly polarized

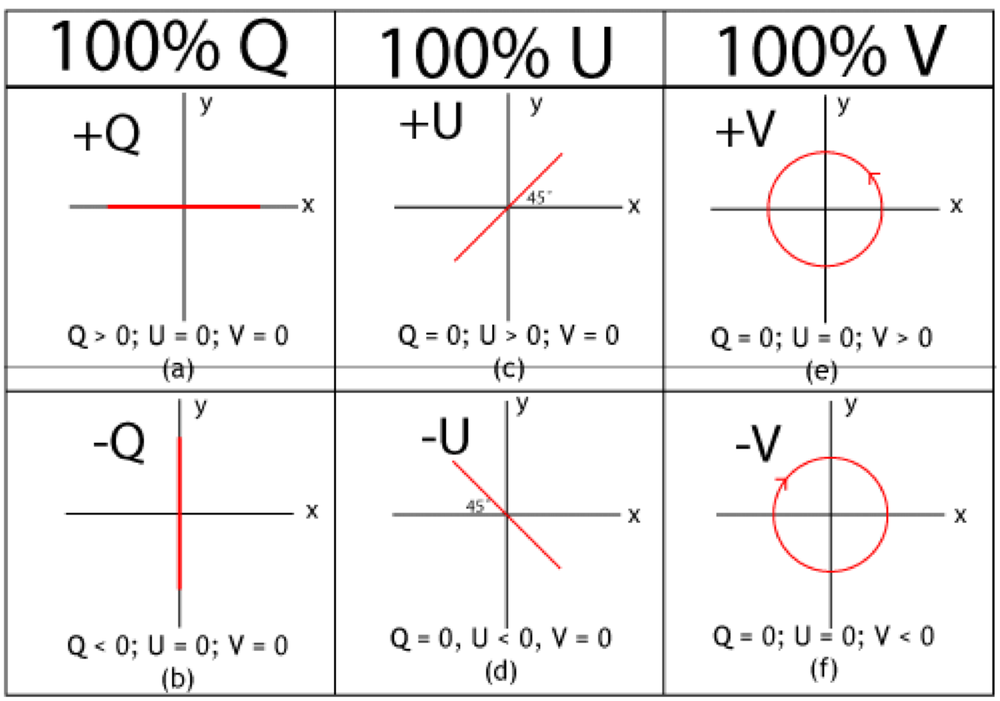
$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ Unpolarized

Alternate explanation

A monochromatic plane wave is specified by its propagation vector, $\vec{k}$, and the complex amplitudes of the electric field, E_1 and E_2, in a basis $(\hat{\epsilon}_1, \hat{\epsilon}_2)$. The pair (E_1, E_2) is called a Jones vector. Alternatively, one may specify the propagation vector, the phase, ϕ, and the polarization state, Ψ, where Ψ is the curve traced out by the electric field as a function of time in a fixed plane. The most familiar polarization states are linear and circular, which are degenerate cases of the most general state, an ellipse.



One way to describe polarization is by giving the semi-major and semi-minor axes of the polarization ellipse, its orientation, and the sense of rotation (See the above figure). The Stokes parameters I, Q, U, and V, provide an alternative description of the polarization state which is experimentally convenient because each parameter corresponds to a sum or difference of measurable intensities. The next figure shows examples of the Stokes parameters in degenerate states.



Definitions

The Stokes parameters are defined by

$$\begin{aligned} I &\equiv \langle E_x^2 \rangle + \langle E_y^2 \rangle \\ &= \langle E_a^2 \rangle + \langle E_b^2 \rangle \\ &= \langle E_l^2 \rangle + \langle E_r^2 \rangle, \\ Q &\equiv \langle E_x^2 \rangle - \langle E_y^2 \rangle, \\ U &\equiv \langle E_a^2 \rangle - \langle E_b^2 \rangle, \\ V &\equiv \langle E_l^2 \rangle - \langle E_r^2 \rangle. \end{aligned}$$

where the subscripts refer to three different bases of the space of Jones vectors: the standard Cartesian basis ($\hat{x}, \hat{y}$), a Cartesian basis rotated by 45° ($\hat{a}, \hat{b}$), and a circular basis ($\hat{l}, \hat{r}$). The circular basis is defined so that $\hat{l} = (\hat{x} + i\hat{y})/\sqrt{2}$.

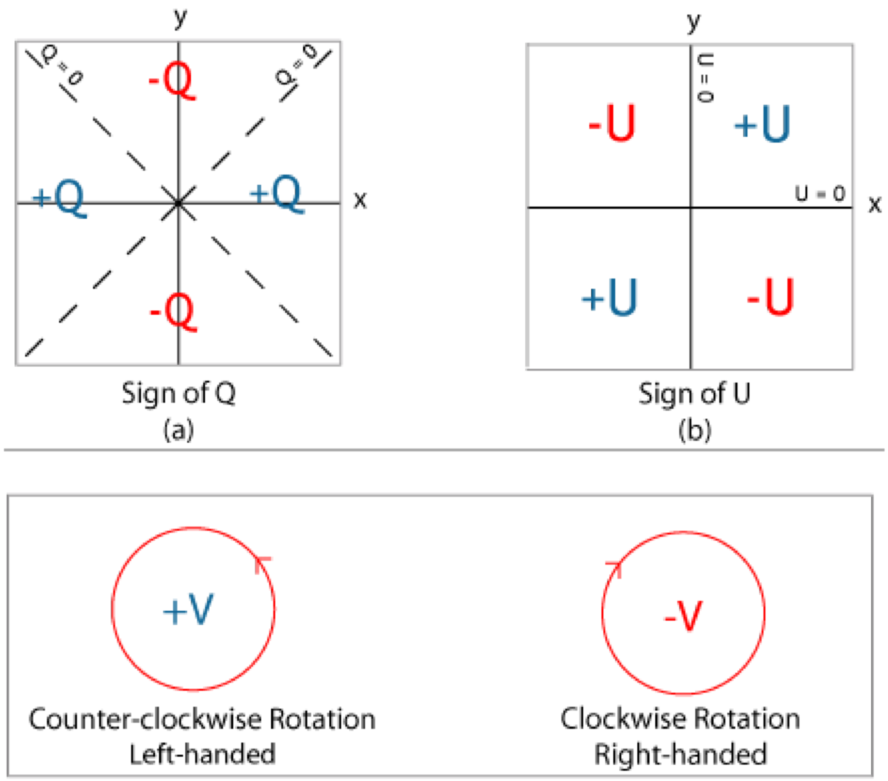
The symbols ⟨·⟩ represent expectation values. The light can be viewed as a random variable taking values in the space C^2 of Jones vectors (E_1, E_2). Any given measurement yields a specific wave (with a specific phase, polarization ellipse, and magnitude), but it keeps flickering and wobbling between different outcomes. The expectation values are various averages of these outcomes. Intense, but unpolarized light will have $I > 0$ but $Q = U = V = 0$, reflecting that no polarization type predominates. A convincing waveform is depicted at the article on coherence.

The opposite would be perfectly polarized light which, in addition, has a fixed, nonvarying amplitude -- a pure sine curve. This is represented by a random variable with only a single possible value, say (E_1, E_2). In this case one may replace the brackets by absolute value bars, obtaining a well-defined quadratic map

$$\begin{aligned} I \equiv |E_x|^2 + |E_y|^2 = |E_a|^2 + |E_b|^2 &= |E_l|^2 + |E_r|^2 \\ Q &\equiv |E_x|^2 - |E_y|^2, \\ U &\equiv |E_a|^2 - |E_b|^2, \\ V &\equiv |E_l|^2 - |E_r|^2. \end{aligned}$$

from the Jones vectors to the corresponding Stokes vectors; more convenient forms are given below. The map takes its image in the cone defined by $|I|^2 = |Q|^2 + |U|^2 + |V|^2$, where the purity of the state satisfies $p = 1$ (see below).

The next figure shows how the signs of the Stokes parameters are determined by the helicity and the orientation of the semi-major axis of the polarization ellipse.



Sign of V
(c)

Representations in fixed bases

In a fixed $(\hat{x}, \hat{y})$ basis, the Stokes parameters when using an *increasing phase convention* are

$$\begin{aligned} I &= |E_x|^2 + |E_y|^2, \\ Q &= |E_x|^2 - |E_y|^2, \\ U &= 2\mathrm{Re}(E_x E_y^*), \\ V &= -2\mathrm{Im}(E_x E_y^*), \end{aligned}$$

while for $(\hat{a}, \hat{b})$, they are

$$\begin{aligned} I &= |E_a|^2 + |E_b|^2, \\ Q &= -2\mathrm{Re}(E_a^* E_b), \\ U &= |E_a|^2 - |E_b|^2, \\ V &= 2\mathrm{Im}(E_a^* E_b). \end{aligned}$$

and for $(\hat{l}, \hat{r})$, they are

$$\begin{aligned} I &= |E_l|^2 + |E_r|^2, \\ Q &= 2\mathrm{Re}(E_l^* E_r), \\ U &= -2\mathrm{Im}(E_l^* E_r), \\ V &= |E_r|^2 - |E_l|^2. \end{aligned}$$

Properties

For purely monochromatic coherent radiation, it follows from the above equations that

$$Q^2 + U^2 + V^2 = I^2,$$

whereas for the whole (non-coherent) beam radiation, the Stokes parameters are defined as averaged quantities, and the previous equation becomes an inequality:[2]

$$Q^2 + U^2 + V^2 \leq I^2.$$

However, we can define a total polarization intensity I_p, so that

$$Q^2 + U^2 + V^2 = I_p^2,$$

where I_p/I is the total polarization fraction.

Let us define the complex intensity of linear polarization to be

$$\begin{aligned} L &\equiv |L|e^{i2\theta} \\ &\equiv Q + iU. \end{aligned}$$

Under a rotation $\theta \to \theta + \theta'$ of the polarization ellipse, it can be shown that I and V are invariant, but

$$\begin{aligned} L &\to e^{i2\theta'} L, \\ Q &\to \mathrm{Re}\left(e^{i2\theta'} L\right), \\ U &\to \mathrm{Im}\left(e^{i2\theta'} L\right). \end{aligned}$$

With these properties, the Stokes parameters may be thought of as constituting three generalized intensities:

$$\begin{aligned} I &\geq 0, \\ V &\in \mathbb{R}, \\ L &\in \mathbb{C}, \end{aligned}$$

where I is the total intensity, $|V|$ is the intensity of circular polarization, and $|L|$ is the intensity of linear polarization. The total intensity of polarization is $I_p = \sqrt{|L|^2 + |V|^2}$, and the orientation and sense of rotation are given by

$$\begin{aligned}\theta &= \frac{1}{2}\arg(L),\\ h &= \operatorname{sgn}(V).\end{aligned}$$

Since $Q = \operatorname{Re}(L)$ and $U = \operatorname{Im}(L)$, we have

$$\begin{aligned}|L| &= \sqrt{Q^2+U^2},\\ \theta &= \frac{1}{2}\tan^{-1}(U/Q).\end{aligned}$$

Relation to the polarization ellipse

In terms of the parameters of the polarization ellipse, the Stokes parameters are

$$\begin{aligned}I_p &= A^2+B^2,\\ Q &= (A^2-B^2)\cos(2\theta),\\ U &= (A^2-B^2)\sin(2\theta),\\ V &= 2ABh.\end{aligned}$$

Inverting the previous equation gives

$$\begin{aligned}A &= \sqrt{\frac{1}{2}(I_p+|L|)}\\ B &= \sqrt{\frac{1}{2}(I_p-|L|)}\\ \theta &= \frac{1}{2}\arg(L)\\ h &= \operatorname{sgn}(V).\end{aligned}$$

Relationship to Hermitian operators and quantum mixed states

From a geometric and algebraic point of view, the Stokes parameters stand in one-to-one correspondence with the closed, convex, 4-real-dimensional cone of nonnegative Hermitian operators on the Hilbert space $\mathbf{C}^2$. The parameter I serves as the trace of the operator, whereas the entries of the matrix of the operator are simple linear functions of the four parameters I, Q, U, V, serving as coefficients in a linear combination of the Stokes operators. The eigenvalues and eigenvectors of the operator can be calculated from the polarization ellipse parameters I, p, ψ, χ.

The Stokes parameters with I set equal to 1 (i.e. the trace 1 operators) are in one-to-one correspondence with the closed unit 3-dimensional ball of mixed states (or density operators) of the quantum space $\mathbf{C}^2$, whose boundary is the Bloch sphere. The Jones vectors correspond to the underlying space $\mathbf{C}^2$, that is, the (unnormalized) pure states of the same system. Note that phase information is lost when passing from a pure state $|\varphi\rangle$ to the corresponding mixed state $|\varphi\rangle\langle\varphi|$, just as it is lost when passing from a Jones vector to the corresponding Stokes vector.

See also

- Mueller calculus
- Jones calculus
- Polarization (waves)
- Rayleigh Sky Model
- Stokes operators
- Polarization mixing

Notes

1. S. Chandrasekhar 'Radiative Transfer, *Dover Publications, New York, 1960, ISBN 0-486-60590-6, page 25*
2. H. C. van de Hulst *Light scattering by small particles*, Dover Publications, New York, 1981, ISBN 0-486-64228-3, page 42

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- E. Collett, *Field Guide to Polarization*, SPIE Field Guides vol. **FG05**, SPIE (2005). ISBN 0-8194-5868-6.
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- William H. McMaster (1961). "Matrix representation of polarization". *Rev. Mod. Phys*. **8**: 33. Bibcode:1961RvMP...33....8M (http://adsabs.harvard.edu/abs/1961RvMP...33....8M). doi:10.1103/RevModPhys.33.8 (https://doi.org/10.1103%2FRevModPhys.33.8).

External links