observed for single straight wires. Projected efficiencies for large wire grids indicate that the banded matrix iterative method may provide solutions for large surfaces at a reasonable cost.

References

[1] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.

[2] V. V. Klyuyev and N. I. Kokovkin–Shcherbak, "On the minimization of the number of arithmetic operations for the solution of linear algebraic systems of equations," (transl. by G. J. Tee), Computer Science Department, Stanford University, Stanford, Calif., Tech. Rep. CS24, 1965.

[3] F. K. Oshiro *et al.*, "Calculation of radar cross section, part II, analytical report," Northrup Corporation AFAL-TR-70-21, Part 2, 1970.

[4] T. R. Ferguson, "Iterative techniques in the method of moments," The EMCAP (Electromagnetic Compatability Analysis Program) Third Quarter Tech. Rep., Part 2, The BDM Corporation, Albuquerque, N. Mex. (RADC-TR-75-121).

[5] ——, "Solution of thin wire moments problems by banded matrix iteration," The EMCAP (Electromagnetic Compatability Analysis Program) Fourth Quarter Tech. Rep., Part 2, The BDM Corporation, Albuquerque, N. Mex. RADC-TR-75-189.

[6] D. J. Rose and R. A. Willoughby, Ed., *Sparse Matrices and Their Applications*. New York: Plenum, 1972.

[7] G. E. Forsythe and C. B. Moler, *Computer Solution of Linear Algebraic Systems*. Englewood Cliffs, N.J.: Prentice-Hall, 1967.

[8] J. R. Stewart, "Application of optimization theory to electromagnetic radiation and scattering," Ph.D. dissertation, Syracuse Univ., Syracuse, N.Y., Jan. 1974.

[9] F. J. Deadrick and E. K. Miller, "Program WAMP (Wire Antenna Modeling Program)," Lawrence Livermore Laboratory, Dec. 1973. (UCID-30084). A Modified Version of a Program Created by MB Associates.

[10] M. T. Ma, *Theory and Application of Antenna Arrays*. New York: Wiley, 1974.

Communications

An Approximate Formula for Calculating the Directivity of an Antenna

CHEN-TO TAI, FELLOW, IEEE, AND CLOVIS S. PEREIRA

Abstract—**An approximate general formula to calculate the directivity of an antenna based upon the E-plane and H-plane patterns is proposed. For narrow beam patterns, the directivity is expressed in terms of the half-power beam widths of the main patterns. The better approximation of the formula presented here over the geometrical mean formula is pointed out.**

The directivity of an antenna is defined as

$$D = \frac{U_{\max}}{\dfrac{1}{4\pi}\displaystyle\iint U(\theta,\phi)\sin\theta\, d\theta\, d\phi} \tag{1}$$

where $U(\theta,\phi)$ denotes the far-zone power density, and it is related to the far-zone electrical field by

$$U(\theta,\phi) = \frac{1}{2Z_0}\left[|E_\theta(\theta,\phi)|^2 + |E_\phi(\theta,\phi)|^2\right]$$

where

$$Z_0 = (\mu_0/\varepsilon_0)^{1/2}.$$

The functions E_θ and E_ϕ which represent the two angular components in a spherical system are, in general, functions of θ and ϕ. In experimental work one often measures the power pattern in two principal planes. These power patterns correspond to

$$|E_\theta(\theta,0)|^2, \qquad \text{the } E\text{-plane pattern}$$

and

$$\left|E_\phi\left(\theta,\frac{\pi}{2}\right)\right|^2, \qquad \text{the } H\text{-plane pattern.}$$

From these two patterns we propose that the directivity of the antenna can be obtained approximately, but quite accurately, using the formula

$$\frac{1}{D} = \frac{1}{2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) \tag{2}$$

where

$$D_1 = \frac{|E_\theta|^2_{\max}}{\dfrac{1}{2}\displaystyle\int_0^\pi |E_\theta(\theta,0)|^2 \sin\theta\, d\theta} \tag{3}$$

$$D_2 = \frac{|E_\phi|^2_{\max}}{\dfrac{1}{2}\displaystyle\int_0^\pi \left|E_\phi\left(\theta,\frac{\pi}{2}\right)\right|^2 \sin\theta\, d\theta}. \tag{4}$$

The simplicity of this approximate formula is that only one-dimensional integrals are involved in evaluating D_1 and D_2. These can be done numerically if the E-plane and the H-plane patterns are available. The formula given by (2) will be referred as the arithmetic-mean formula. The expression for D_1 defined by (3) corresponds to the directivity of an antenna with a rotationally symmetrical pattern $|E_\theta(\theta,0)|^2$ and that for D_2 with a rotationally symmetrical pattern $|E_\phi(\theta,(\pi/2)|^2$. For uniform arrays of short dipoles operated as broadside arrays or as end-fire arrays it can be shown that the arithmetic-mean formula is exact. For arrays made of half-wave dipoles there is sufficient evidence that the formula is quite accurate.

For antennas with a narrow beam pattern it is desirable to relate D_1 and D_2 in terms of the half-power beamwidth of the E-plane and the H-plane pattern, hereby denoted by θ_1 and θ_2. An approximate expression for D_1 in terms of θ_1 is

$$D_1 \simeq 16 \ln 2/\theta_1{}^2$$

and similarly

$$D_2 \simeq 16 \ln 2/\theta_2{}^2$$

Manuscript received August 18, 1975; revised October 13, 1975. This work was supported in part by the National Science Foundation under Grant GK-36867.

C. T. Tai is with the Instituto de Pesquisas Espaciais, São Paulo, Brasil, on leave from the Department of Electrical and Computer Engineering, University of Michigan, Ann Arbor, MI 48104.

C. S. Pereira is with the Instituto de Pesquisas Espaciais, São Paulo, Brasil.

hence

$$D \simeq 32 \ln 2/(\theta_1^2 + \theta_2^2). \quad (5)$$

These expressions are obtained by considering the asymptotic expression for the directivity of an antenna with a rotationally symmetrical power pattern of the form $U(\theta) = \cos^m \theta$, for $\pi/2 \geq \theta \geq 0$, and $U(\theta) = 0$, for $\pi > \theta > \pi/2$ with very large value of m. Alternative formulas can be derived by using the asymptotic expressions generated by Legendre polynomials or Chebyshev polynomials. This subject will be discussed in detail on another occasion.

It is recalled that an approximate expression for D proposed by Kraus [1] for a narrow beam pattern is

$$D_K \simeq \frac{4\pi}{\theta_1\theta_2}. \quad (6)$$

Consider the same power pattern $\cos^m \theta$ defined above. For $m = 2$, the exact value of D is six. Equation (5), with $\theta_1 = \theta_2$, yields $D = 4.62$ while (6) yields $D_K = 5.09$. For $m = 100$, the exact value of D is 202 while the approximate values obtained by (5) and (6) are, respectively, 206 and 233. It would be interesting to verify experimentally which formula gives a better overall approximation, particularly for asymmetrical patterns.

It should be mentioned that an approximate formula in the form of $D = \sqrt{D_1 D_2}$ was once proposed by Boithias *et al.* [2]. Their formula does not appear to have the analytical foundation as compared to ours (2) when applied to uniform arrays of dipoles. For the case of a short horizontal electrical dipole we have

$$\bar{E} = E_0[\cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}]$$

hence

$$D_1 = 3 \qquad D_2 = 1$$

thus the arithmetic-mean formula gives $D = \frac{3}{2}$, which is exact, while the geometric-mean formula yields $D = \sqrt{3} = 1.73$, an error of 15 percent.

Acknowledgment

A valuable comment on our original manuscript prompted us to give a better presentation of the approximation relating the directivity and the beamwidth for narrow beam patterns. The reviewer's careful reading of our manuscript is very much appreciated.

References

[1] John D. Kraus, *Antennas*. New York: McGraw-Hill, 1950, p. 25.
[2] L. Boithias, R. Bété, and C. Mathieu, "Le centre d'études d'antennes du CNET a la Turbié," L'écho des Recherches, July 1967.

Note added in Proof: In a private communication with Prof. John D. Kraus, he called our attention to the general formula relating the directivity and the half-power beamwidth described in his book *Radio Astronomy* (New York: McGraw-Hill, 1966, p. 158). In the context of that general formula, the pattern shape factor which we have derived has the value of $\pi/4 \ln 2$ or 1.13. It is very close to the maximum value prescribed by Kraus (1.05 ± 0.05). The authors wish to thank Prof. Kraus for his comment.

Transient Fields of Small Loop Antennas

KARL JÖRG LANGENBERG

Abstract—**A rather simple and general analysis allows us to calculate directly the transient response of many different radiators of practical interest in the time domain without the knowledge of the corresponding time harmonic solution. For example, we present exact expressions for the transient near- and far-field around small circular current loops involving arbitrarily time-varying excitation. The resulting convolution integrals have been numerically evaluated for step function currents with finite rise time, thus showing the influence of finite loop radius on the radiation field in comparison with the ideal magnetic dipole.**

Manuscript received June 21, 1975; revised September 9, 1975.
The author is with Fachrichtung Elektrotechnik, Universität des Saarlandes, Saarbrücken, Germany.

I. Introduction

A recent paper by G. Franceschetti and C. H. Papas [1] presents a number of very interesting procedures and results—besides numerous references—relating to the computation of transient radiation from elementary sources. Here, we take one of their examples—the small circular loop antenna—to give a new and quite simple approach to the handling of certain structures radiating arbitrarily in time. The method applies an idea credited to Cagniard [2] which has been used to solve seismic and electromagnetic pulse problems [3], [4]. It allows us to extend the results of Franceschetti and Papas to the near-field radiation zone leading to exact expressions for the components of the electromagnetic field without the knowledge of the corresponding time harmonic solution. This is especially interesting in those cases where this solution is rather complicated as for large loop antennas with sinusoidal current distribution [5]. The terms "small" or "large" apply to the vT space scale, T being the rise time of a step function current and v the light phase velocity of the isotropic, homogeneous, and nondispersive medium surrounding the antenna. The consideration of finite rise times instead of ideal step functions is of great importance in the physical discussion of the results, because the assumption of constant current over all the small loop is only fulfilled if $a \ll vT$, a being the loop radius; that is to say the results based on ideal step functions only comply to the limiting case $a \to 0$, i.e., to the ideal magnetic dipole. For finite $a > 0$ Franceschetti's and Papas' statement, that the radiated far-field of the small loop is *not* proportional to the second time derivative of the current, is only of mathematical interest because an essential physical assumption is not fulfilled. Our numerical results show clearly that up to a certain approximation the radiated far-field of small loop antennas *is* proportional to the second time derivative of the current; a finite loop radius $a > 0$ results only in small modifications of the transient near- and far-field of the ideal magnetic dipole. This might be of interest for the investigations of Harmuth and Fralick [6], [7] concerning the radiation of Walsh-waves from small loop antennas.

II. Mathematical Formulation of Problem and Setup of Solution

The electromagnetic field of a given current density $J(r,t)$, r being the vector of position and t the time, can be determined from the retarded vector potential $A(r,t)$, where

$$A(r,t) = \frac{\mu_0\mu}{4\pi}\int_V \frac{J\left(r',t - \dfrac{|r - r'|}{v}\right)}{|r - r'|}\,dV'. \quad (1)$$

The vector r' denotes the position in the source region V; $v = (\varepsilon_0\varepsilon\mu_0\mu)^{-1/2}$ is the light phase velocity, $\varepsilon_0\varepsilon$ the permittivity, and $\mu_0\mu$ the permeability of the isotropic, homogeneous, and nondispersive medium surrounding V. Laplace transforming equation (1), s being the variable in the transform space, yields

$$a(r,s) = \frac{\mu_0\mu}{4\pi}\int_V j(r',s)\,\frac{e^{-s(|r-r'|/v)}}{|r - r'|}\,dV'. \quad (2)$$