representation and energy concepts,[6] can also be associated analogously with the optimal-control problem, and the piecewise solution of the network problem can be related to sequential least-squares estimation.†

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MAXIMUM GAIN OF YAGI–UDA ARRAYS

Indexing term: Antenna arrays

Numerical optimisation techniques have been used to find the maximum gain of some specific parasitic arrays. The gain of an array of infinitely thin, equispaced dipoles loaded with arbitrary reactances has been optimised. The results show that standard travelling-wave design methods are not optimum. Yagi–Uda arrays with equal and unequal spacing have also been optimised with experimental verification.

For antenna arrays with driven elements, the maximum achievable gain is well known in many cases of practical interest (see e.g. Lo, Lee and Lee[1]). However, with only one driven element and the remaining elements parasitic, the maximum gain is only known for arrays with very few elements.[2, 3]

Parasitic arrays with many elements are usually designed on the basis of surface-wave antenna theory, notably in the work of Ehrenspeck and Poehler,[4] who found optimum phase velocities as functions of antenna length. It is characteristic of this approach that it considers only homogeneous structures, since all the parasitic elements are equal, except perhaps for a slight taper to reduce reflections from the end of the structure. The current distribution along the travelling-wave array usually has a maximum at the feeding element and a relatively constant, somewhat smaller value over the parasitic part. This is in contrast to the optimum distribution[1] for driven arrays, where the magnitude of the current should be largest at the centre of the array and taper off to a smaller value at the ends of the array. This latter distribution is closer to a standing-wave distribution than a travelling-wave distribution, and suggests that maximum gain is achieved by an inhomogeneous structure, tapered in such a way that reflections from the end are enhanced. Furthermore, the driven element should be close to the centre of the array rather than at one end. It is precisely such a structure which has been found by computer optimisation methods, the results of which are presented in this letter.

The approach is to use a strictly numerical technique, available computer optimisation procedures[5] being used to find the maximum gain. Three particular cases of linear parasitic arrays with one element excited are considered. The first consists of infinitely thin equispaced halfwave dipoles, loaded at their centres with arbitrary reactances which are the independent variables. The second and third consist of cylindrical dipoles, where the independent variables are the element lengths and the spacing between elements; the elements are equispaced in the second case and have variable spacing in the third. In all cases, the current distribution along the elements has been assumed to be sinusoidal,

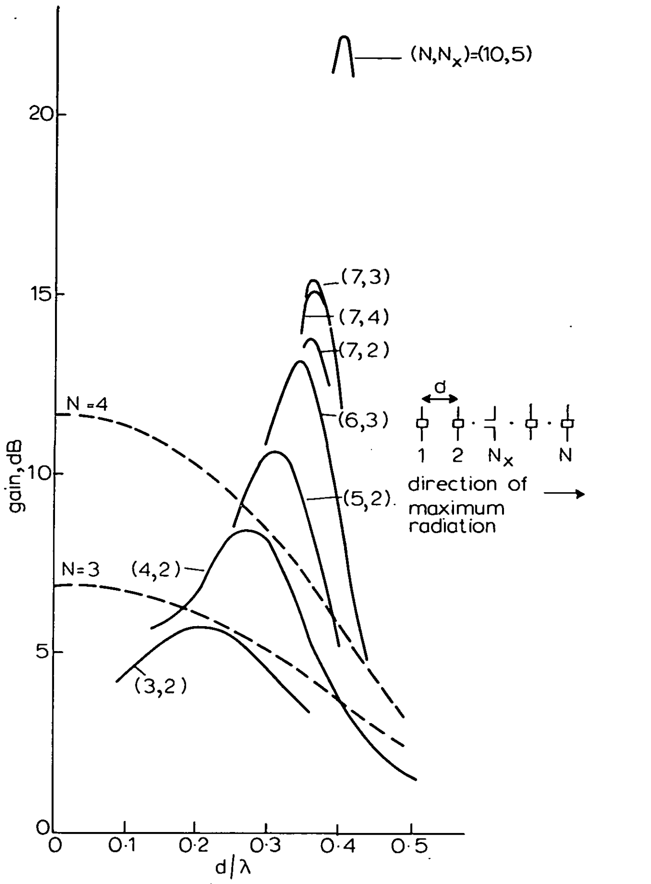

Fig. 1A *Maximum relative endfire gain*
——— as a power ratio for arrays of N relatively thin, loaded halfwave dipoles as a function of element spacing d. Element N_x is driven and the remaining elements are parasitic
– – – maximum relative endfire gain of an N-element array with all elements driven
(N, N_x) refer to the number N of elements and the position of the driven element

and only element lengths around half a wavelength have been considered.

The method has been to excite one element and then find the currents in all the elements by inversion of the impedance matrix. The self and mutual impedances have been calculated by means of the induced-e.m.f. method. After determination of the currents, all other relevant antenna parameters such as gain, impedance etc. are easily found.

Maximum relative power gain in the endfire direction is shown in Fig. 1A for an N-element equally spaced array of loaded halfwave dipoles with element number N_x excited. The extreme element to the left is number one and the direction of maximum radiation is to the right. The numbers in parentheses attached to the curves are (N, N_x). The dotted curves show the maximum gain when all the elements are driven, clearly showing the supergain effect when d/λ tends to zero. These curves are taken from Reference 6, Pt. I, p. 198. The main difference is that, for the parasitic case, there is an optimum spacing which increases with N. The curves shown are for optimum values of N_x, except for $N = 7$, where it is shown how the maximum gain varies with N_x. Clearly, for N large, the driven element is near the centre of the array. It is worth noting that, at the optimum value of spacing, there is only a small difference between the maximum gain for the parasitic and the driven arrays.

Fig. 1B shows the optimum distribution of loading reactances corresponding to the maximum gain values of Fig. 1A. The reactances are normalised with respect to the radiation resistance of a single element. We note the non-uniform distribution with the largest absolute value of the reactance near the centre of the array, quite contrary to the usual practice. In terms of a local phase velocity, the local wave is fastest near the centre and slowest near the edges, indicating the presence of a reflected wave, since the reflection from the end is larger for a slower wave.

The maximum gains relative to a halfwave dipole are shown

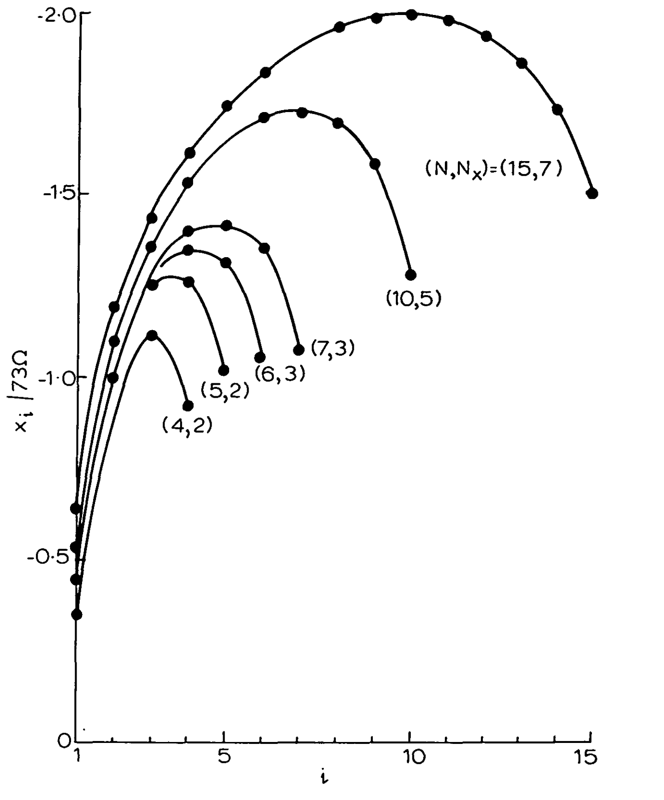

Fig. 1B *Optimum distribution of loading reactances X_i for equispaced, infinitely thin, loaded halfwave dipoles as a function of the element number i*

The excited element is omitted since it is not loaded. (N, N_x) refer to the number N of elements and the position of the driven element

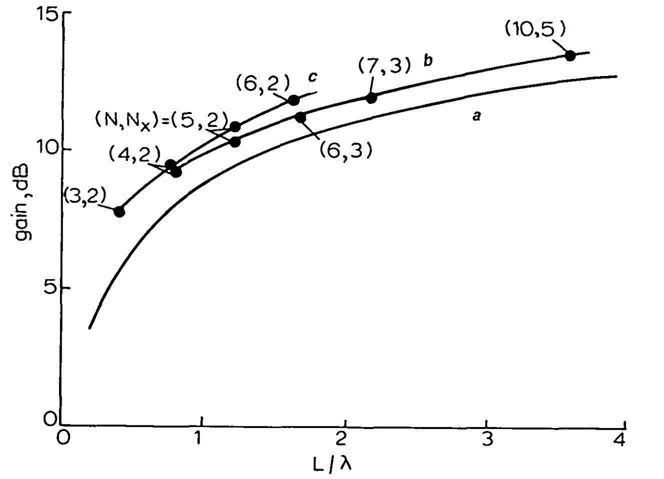

Fig. 2 *Maximum gain of Yagi–Uda arrays as a function of total array length L*

a Experimental results obtained by Ehrenspeck and Poehler[4]
b Equispaced array of infinitely thin, loaded halfwave dipoles, and cylindrical dipoles with variable element lengths
c Array with unequal spacing
(N, N_x) refer to the number N of elements and the position of the driven element

in Fig. 2. The numbers at the points are again (N, N_x). Curve *a* gives the experimental results of Ehrenspeck and Poehler.[4] Curve *b* shows the results of Fig. 1A, and also the results of the second case with an array of cylindrical dipoles with variable lengths but constant spacing. For a given length the gain increase is moderate, but it should be noted that the results of curve *b* are for a minimum number of elements. Curve *c* covers the case of nonuniform spacing, which gives an additional gain increase. By translating the results of Fig. 2 into a linear scale, the following formulas for power gain above a halfwave dipole can be derived:

$$\text{gain} = 5{\cdot}5 + 3{\cdot}5\frac{L}{\lambda}\left(\frac{L}{\lambda} + 1{\cdot}5\right) \quad \text{(Ehrenspeck and Poehler)}$$

$$\text{gain} = 4{\cdot}1 + 5{\cdot}4\frac{L}{\lambda} \quad \text{(equispaced array)}$$

$$\text{gain} = 3{\cdot}0 + 7{\cdot}0\frac{L}{\lambda} \quad \text{(nonequispaced array)}$$

where L is the total antenna length. Thus it appears that this reflection-type Yagi–Uda antenna has, asymptotically, twice

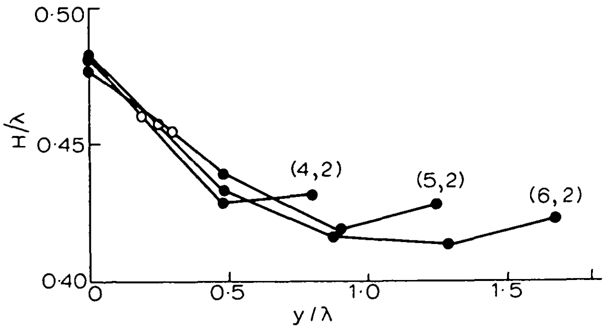

Fig. 3 *Element lengths H and element positions Y for optimum arrays of unequal spacing*

Driven element positions $a/\lambda = 0{\cdot}01$, where a is radius of elements. (N, N_x) refer to the number N of elements and the position of the driven element

the gain of the conventional travelling-wave type, or, expressed differently, for the same gain it only needs half the length. This is somewhat similar to the backfire antenna which, however, requires a much larger extent transverse to the antenna axis.

Finally, Fig. 3 shows some examples of the dimensions of optimum nonuniform arrays for $a/\lambda = 0{\cdot}01$, where a is the radius of the elements. With some slight corrections of the element lengths, the gain values have been confirmed experimentally to within 0·2 dB. The method is being extended to include constraints on Q factor and radiation pattern. Additional information may be found in Reference 7.

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