

Helical Beam Antennas for Wide-Band Applications*

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Summary—The helical beam antenna has inherent broad-band properties. Over a wide frequency band the pattern shape, circularity of polarization, and terminal impedance are relatively stable. Measured performance data are presented for a medium-gain helical beam antenna of optimum dimensions with a bandwidth of about 1.7 to 1. A high-gain broadside array of four such helices is described. Other wide-band applications of helical beam antennas, including omnidirectional types, are also discussed.

INTRODUCTION

A HELIX WITH a circumference of about one wavelength can radiate as a beam antenna.¹ Radiation is maximum in the direction of the helix axis and is circularly polarized, or nearly so. This mode of radiation, called the axial or beam mode, may persist over a wide frequency range.² In footnote reference 2 basic phenomena associated with the beam mode are described, and a method is developed for calculating the radiation patterns. Impedance measurements³ reveal that in the frequency range of the beam mode the terminal impedance is relatively constant and equal to a resistance of about 130 ohms for typical helices. These properties all combine to make the helical beam antenna particularly well suited for wide-band applications.

The dimensions providing the most uniform radiation and impedance characteristics over the greatest frequency range will be referred to as "optimum" dimensions. It is the purpose of this paper to consider the design and performance of such an optimum helix. This subject is not treated in the previous papers. Operation of this helix in multiple to provide a high-gain beam is also considered, as are other wide-band applications of helical beam antennas.

It should be mentioned that the beam mode of radiation is but one of many modes in which a helix may radiate.⁴ The characteristics of not only the beam mode, but also other modes are considered in detail in another paper.⁵ The present paper deals only with the beam mode of radiation as produced by uniform helices of circular or square cross section.

One of the outstanding characteristics of the beam mode of radiation of a helical antenna is the ease with which circularly polarized radiation is obtained. The

beam mode of radiation can be readily produced by operating the helix with a ground plane, the combination being energized by a coaxial transmission line as

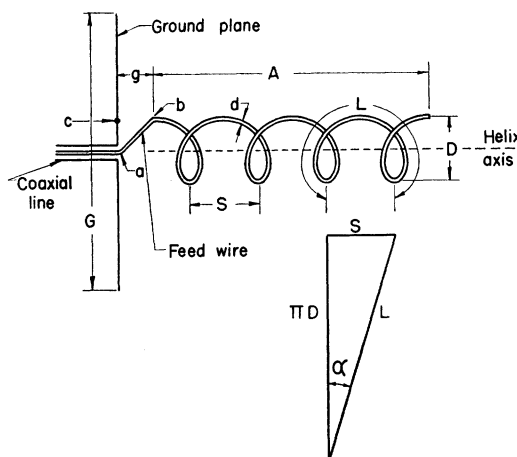


Fig. 1—Helix and associated dimensions.

in Fig. 1. The outer conductor terminates in the ground plane and the inner conductor connects to the end of the helix.

The following symbols are used to describe the helix and ground plane (see Fig. 1):

D = diameter of helix

S = spacing between turns (center-to-center)

α = pitch angle = $\arctan S/\pi D$

L = length of one turn

n = number of turns

A = axial length = nS

d = diameter of helix conductor

g = distance of helix proper from ground plane

G = ground plane diameter.

If one turn of a helix is unrolled on a flat plane, the circumference (πD), spacing (S), turn length (L), and pitch angle (α) are related by a triangle as shown in Fig. 1.

In Fig. 1 the coaxial line is coincident with the helix axis and the feed wire (between a and b) lies in a plane through the helix axis. Beyond point b the conductor lies in the surface of the imaginary helix cylinder. This is the helix proper of axial length A . The component of the feed wire length parallel to the axis is g . In the helices to be described, g is equal to about $S/2$. The antenna terminals are considered to be at the point of connection with the coaxial line and all impedances are referred to this point (a). It is sometimes more convenient to place the coaxial-line terminals at a point which is $D/2$ from the axis as indicated by the point c in Fig. 1. However, in the antennas described herein the coaxial-line terminals are coincident with the helix axis.

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¹ J. D. Kraus, "Helical beam antenna," *Electronics*, vol. 20, pp. 109-111; April, 1947.

² J. D. Kraus and J. C. Williamson, "Characteristics of helical antennas radiating in the axial mode," *Jour. Appl. Phys.*, vol. 19, pp. 87-96; January, 1948.

³ O. J. Glasser and J. D. Kraus, "Measured impedances of helical beam antennas," *Jour. Appl. Phys.*, vol. 19, pp. 193-197; February, 1948.

⁴ H. A. Wheeler, "A helical antenna for circular polarization," *PROC. I.R.E.*, vol. 35, pp. 1484-1488; December, 1947. (This paper discusses the "normal" mode of radiation.)

⁵ J. D. Kraus, "The helical antenna," to be published.

As the frequency varies, the helix diameter D_λ and spacing S_λ in free-space wavelengths change, but the pitch angle remains constant. The relation of D_λ , S_λ , and α as a function of frequency is conveniently illustrated by a diameter-spacing chart as in Fig. 2. The dimensions of any uniform helix are defined by a point on the chart. Let us consider a helix of pitch angle equal to 10 degrees. At zero frequency, $D_\lambda = S_\lambda = 0$. With increase in frequency, the co-ordinates (S_λ , D_λ) of the point giving the helix spacing and diameter increase, but their ratio is constant so that the point moves along the constant-pitch-angle line for 10 degrees. Designating the lower and upper frequency limits of the frequency range

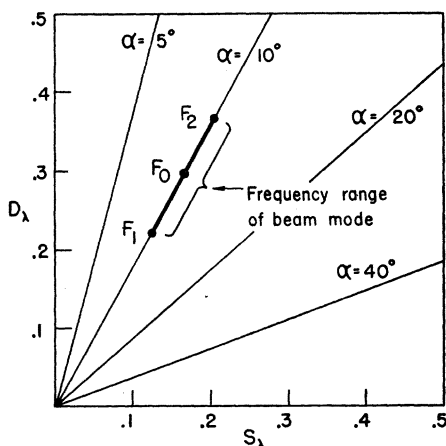


Fig. 2—Diameter-spacing chart for helices showing range of dimensions associated with a frequency band.

of the beam mode as F_1 and F_2 , respectively, the corresponding range in spacing and diameter is given by a line between the points for F_1 and F_2 on the 10-degree line. The center frequency of the range is F_0 and is taken arbitrarily such that $F_0 - F_1 = F_2 - F_0$ or $F_0 = (F_1 + F_2)/2$. The dimensions of the helices to be described are given in free-space wavelengths at this center frequency F_0 .

DETERMINATION OF AN OPTIMUM HELIX

Pattern and impedance data are given in footnote reference 3 for helical antennas of fixed physical length, but of pitch angles ranging from 6 to 24 degrees. The antennas are about 1.6 wavelengths long at the center frequency of the beam mode range with half-power beam widths at this frequency of about 40 degrees. An antenna of this size and directivity is suitable for many high-frequency and microwave transmitting and receiving applications.

An optimum helix may be determined by comparing pattern and impedance data taken from footnote references 2 and 3 on a D - S chart, as in Fig. 3. The pattern contour in Fig. 3 indicates the approximate region of satisfactory patterns. A satisfactory pattern is considered to be one with a major lobe in the axial direction and with relatively small minor lobes. Inside the pattern contour of Fig. 3 the patterns are of this type, and have

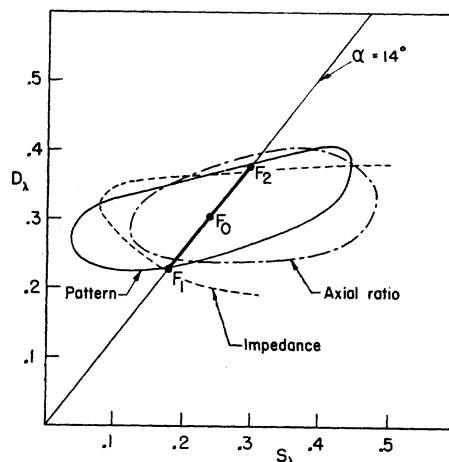


Fig. 3—Diameter-spacing chart for helices with contours showing regions of stable pattern shape, and terminal impedance, and of low axial ratio, for helices of fixed physical length.

beam widths of from 30 to 60 degrees. Inside the impedance contour in Fig. 3 the terminal impedance is relatively constant (between 100 and 150 ohms), and is nearly a pure resistance. This region is the "impedance plateau" of footnote reference 3. A third contour in Fig. 3 is for the axial ratio measured in the direction of the helix axis.⁶ Inside this contour⁷ the axial ratio is less than 1.25. From a consideration of the three contours it is apparent that too small or too large a pitch angle is undesirable. An "optimum" pitch angle appears to be about 14 degrees. Since the properties change slowly as a function of α in the vicinity of 14 degrees, there is nothing critical about this value. In fact, the properties of helices of pitch angles of 14 ± 2 degrees differ but little. Referring to Fig. 3, a line for $\alpha = 14$ degrees is indicated with upper and lower frequency limits for satisfactory operation. Although the exact location of these limits is arbitrary, it is relatively well defined by the close bunching of the contours for the three properties (pattern, axial ratio, and impedance) near the frequency limits. The frequency range between F_1 and F_2 is 1.67 to 1 ($F_2/F_1 = 1.67$). Although the optimum pitch angle of 14 degrees associated with this frequency range applies specifically to a helix with an over-all axial length ($A + g$) of about 1.65 wavelengths and a conductor diameter of 0.017 wavelength at the center frequency, it is probable that 14 degrees is close to optimum for helices that are considerably shorter or longer, or are of somewhat different conductor diameter.

Referring to Fig. 1 and taking $g = S/2$, we have

$$A + g = S(n + 1/2), \text{ or} \\ n = \frac{A + g}{S} - 1/2. \quad (1)$$

⁶ Axial ratio is defined as the ratio of the major to minor axes of the polarization ellipse. It is one for circular polarization and infinite for linear polarization.

⁷ From data by J. C. Williamson, "An investigation of some radiation characteristics of helical antennas," master's thesis, Ohio State University, 1947.

Since $S=0.24$ wavelength at the center frequency for the 14-degree helix, the number of turns (n) from (1) is 6.4. Taking the nearest integral number gives $n=6$. Thus, the helix chosen as an optimum for general-purpose wide-band applications has 6 turns and a pitch angle of 14 degrees.

PERFORMANCE OF OPTIMUM HELIX

A 6-turn 14-degree right-handed helix was constructed and its characteristics measured. Fig. 4 is a photograph of the antenna, and Fig. 5 gives details of the electrical and mechanical construction.⁸ The overall axial length ($A+g$) of the antenna is 118 cm, and the ground-plane diameter (G) is 60 cm. The center frequency is 400 Mc with $F_1=300$ and $F_2=500$ Mc. The mechanical arrangement suggests merely one possible method of mounting the antenna. The helix and ground-

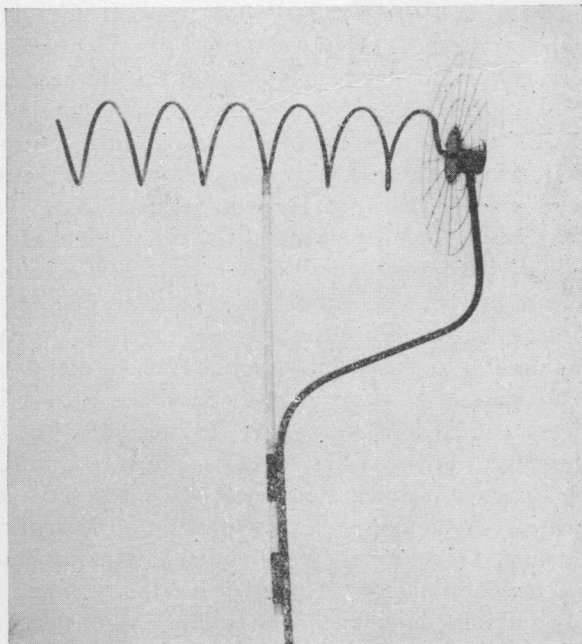


Fig. 4—Optimum helical beam antenna with ground plane. The helix has 6 turns and a pitch angle of 14 degrees. The diameter of the helix is 23 cm and the center frequency 400 Mc.

plane assembly is supported by a single 1-inch-od vertical pipe. The ground plane of sixteen radial and four concentric wires is light in weight and offers little wind resistance. All ground-plane joints are soldered. The helix is of $\frac{1}{2}$ -inch-od tubing and is supported by two insulators, one at the ground plane and one near the middle. The nearly complete absence of dielectric material, except air, gives a more constant terminal resistance as a function of frequency than when the helix is wound, for example, around several dielectric rods as a support. The feed wire is a continuation of the helix conductor and is horizontal. The antenna connects to a

⁸ The helix in Fig. 1 is diagrammatic. Although the helix in Fig. 5 is more nearly a true picture, some liberties have been taken to simplify the drafting.

53-ohm coaxial line through a two-section wide-band transformer. A 130-ohm transmission line connected directly to the antenna terminals would provide an

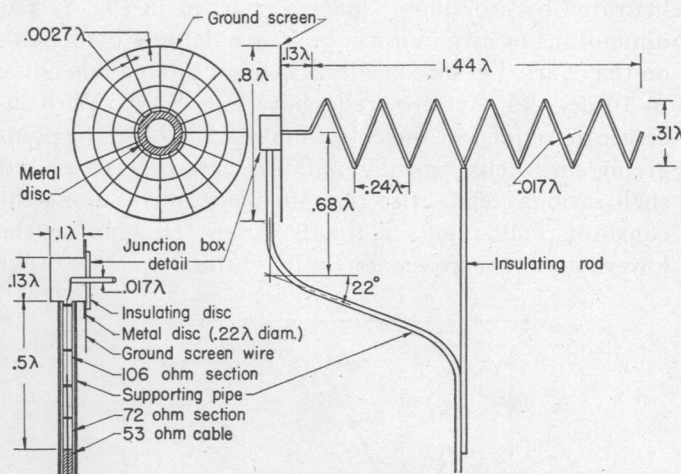


Fig. 5—Details of electrical and mechanical construction of the helical beam antenna shown in Fig. 4. Dimensions are in free-space wavelengths at the center frequency.

ideal method for energizing a helical beam antenna.⁹ To operate the antenna with a commercially available type, such as standard 50- to 53-ohm cable, requires a transformer between the antenna and the cable for maximum power transfer. Each transformer section is about one-quarter wavelength long at the center frequency. The section adjacent to the antenna terminal has a characteristic impedance of 106 ohms, and the other a characteristic impedance of 72 ohms. These impedances differ somewhat from the optimum values for such a transformer, but were chosen as the best compromise with the wire and tubing sizes available. Actually, no dimensions shown in Fig. 5 are critical.

The measured radiation (electric field) patterns of the 6-turn 14-degree helix are presented in Fig. 6 for frequencies from 225 to 600 Mc. The solid curves show the patterns of the horizontally polarized component, and the dashed curves the patterns of the vertically polarized component. All patterns are adjusted to the same maximum value. Referring to the helix in Fig. 6 (lower right), the patterns are in the plane of the page, the horizontal component being parallel to, and the vertical component normal to, the page.

It is evident from these patterns that the axial mode of radiation occurs for frequencies between about 290 and 500 Mc. This mode is characterized by patterns with a large major lobe in the axial direction and relatively small minor lobes. At frequencies less than 290 Mc, the maximum radiation is, in general, not in the axial direction and minor lobes, although few in number, are large. At frequencies above 500 Mc, the minor lobes become both large and numerous.

⁹ A 125-ohm cable designated RG-63/U is now manufactured by the Federal Telephone and Radio Corporation.

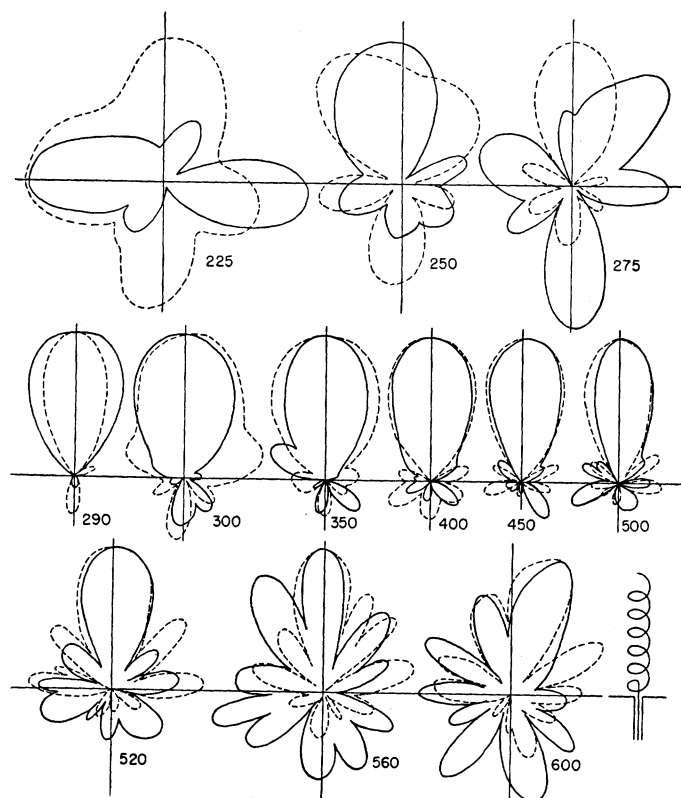


Fig. 6—Measured azimuthal electric field patterns of the 6-turn, 14-degree helix shown in Fig. 4. The solid patterns are for the horizontally polarized, and the dashed patterns for the vertically polarized field component. Between 290 and 500 Mc the patterns are characteristic of the fundamental beam mode of radiation.

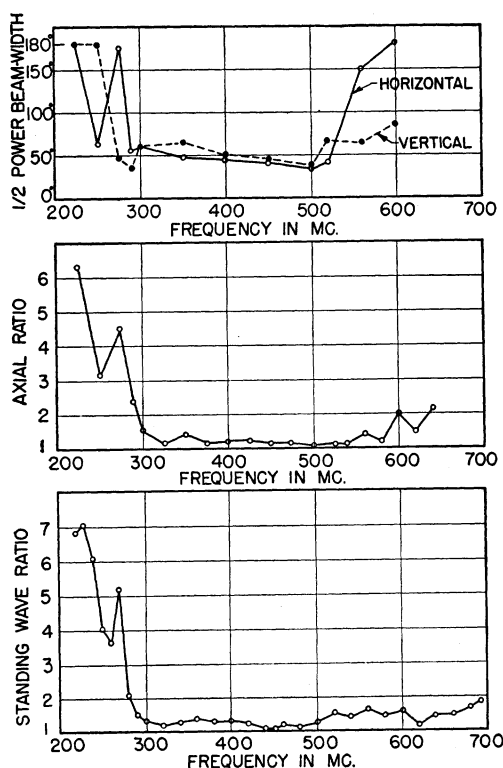


Fig. 7—Summary of measured performance of the 6-turn, 14-degree helix shown in Fig. 4. Half-power beam widths of the horizontal and vertical electric field components, axial ratio in the direction of the helix axis, and standing-wave ratio on a 53-ohm line are presented as a function of frequency.

Pattern, polarization, and impedance properties of the antenna are summarized in Fig. 7. In the uppermost section of the figure, the half-power beam width of the patterns for both the vertical and horizontal components are presented as a function of frequency in megacycles. These data are taken from Fig. 6.

The half-power beam width is taken between half-power points, regardless of whether these occur on the major lobe or on minor lobes. This definition is arbitrary, but is convenient to take into account a splitting up of the pattern into many lobes of large amplitude. Beam widths of 180 degrees or more are arbitrarily plotted as 180 degrees. Curves for the axial ratio and standing-wave ratio (SWR) are given in the lower sections of the figure. The standing-wave ratio was measured on the 53-ohm line about 9 meters from the antenna terminals.

Between 300 and 500 Mc the half-power beam width ranges from about 60 to 40 degrees. Based on pattern integration, the directivity or power gain of the 6-turn 14-degree helix over a nondirectional circularly polarized antenna varies from about 11 (10.4 db) at 300 Mc to about 25 (14 db) at 500 Mc. Between 300 and 500 Mc the axial ratio in the direction of the helix axis varies from 1.05 to 1.5, being less than 1.2 for most of the range. From a practical standpoint, this represents a relatively small deviation from circular polarization. Between 300 and 500 Mc the SWR varies from 1.03 to 1.4. Considered altogether, these pattern, polarization, and impedance characteristics represent remarkably good performance over a wide frequency range, especially since the antenna is merely a simple geometric form with no compensating devices attached except a transformer to convert the 130-ohm terminal resistance to the value of the transmission line (53 ohms).

HIGH-GAIN ARRAYS USING HELICAL BEAM ANTENNAS

Circularly polarized antennas of considerably greater directivity than is provided by the single 6-turn 14-degree helix described in the preceding section can be obtained with helical beam antennas in a variety of arrangements. Four methods are illustrated in Fig. 8. Thus, as suggested in Fig. 8(a), the number of turns might be increased. However, any considerable improvement in directivity would require a very large increase in the length. For example, the axial length A of the 6-turn 14-degree helix is 1.44 wavelengths at the center frequency, and its directivity or gain over an isotropic circularly polarized antenna is about 12 db at this frequency. To increase the gain by 10 db, or to 22 db, the helix length must be multiplied by a large factor so that the total length is of the order of 20 wavelengths. Since a broadside arrangement of much smaller maximum dimensions could produce the same gain, an antenna of such length would be impractical for most applications. The underlying reason for this is not a

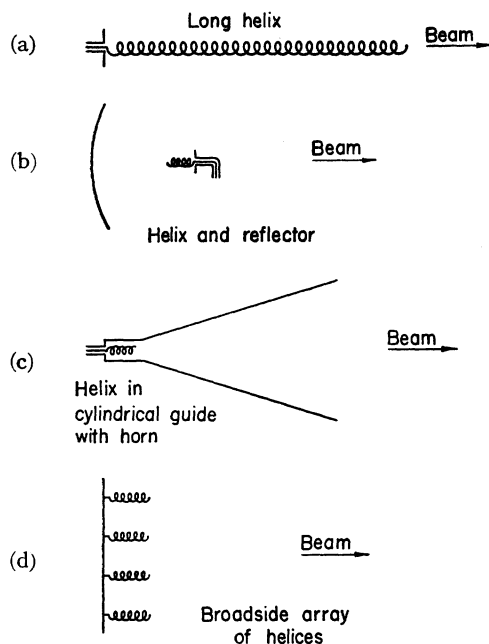


Fig. 8—High-gain antenna systems using helical beam antennas.

characteristic that is peculiar to long helical antennas, but is rather a fundamental property of all long end-fire arrays. Another disadvantage of a very long helix is that no control is afforded over the size of the minor lobes. Thus, while longer helices than the 6-turn 14-degree type described above may be used to provide a moderate increase in directivity, a more practical trend in design for very high gains appears to be toward a broadside type of arrangement. This might take the form of one of the systems suggested in Fig. 8(b), (c), and (d). In Fig. 8(b) a helical beam antenna acts as the primary antenna to "illuminate" a sheet-metal reflector of parabolic or other shape. By adjustment of the illumination of the reflector by the primary helical beam antenna, control of both the beam shape and the size of minor lobes is afforded. Referring to the example considered above, gains of the order of 22 db would be possible with a parabolic reflector of circular section of about 5 wavelengths diameter, and greater gains with larger diameters.

In Fig. 8(c) a helical beam antenna is used to excite a circularly polarized TE_{11} mode in a cylindrical waveguide connected to a cylindrical horn. The area of the aperture of the horn for a given gain will be approximately the same as for the reflector arrangement.

In Fig. 8(d) a broadside array of helices is suggested as an arrangement for obtaining a circularly polarized antenna with high gain. As a specific example of this type, an array of four helices is described in the next section.

FOUR-HELIX BROADSIDE ARRAY

Fig. 9 gives the dimensions for a broadside array of four helical beam antennas. Each helix is of the 6-turn 14-degree type described above. Dimensions are given

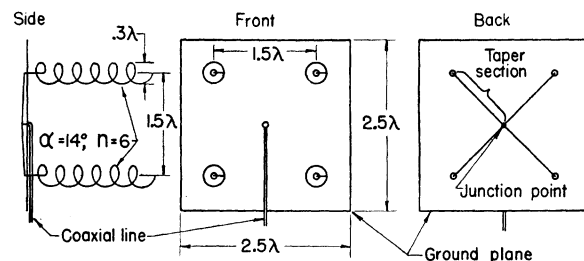


Fig. 9—Constructional details for broadside array with four 6-turn, 14-degree helices. Dimensions are in free-space wavelengths at the center frequency.

in free-space wavelengths at the center frequency. The helices are mounted on a flat square ground plane of 2.5 by 2.5 wavelengths. All helices are oriented in the same manner, and are energized with equal, in-phase voltages. The helices are symmetrically placed and spaced 1.5 wavelengths between centers. All of the helices are wound in the same direction, and the radiation is circularly polarized. If two of the helices were wound left-handed and the other two right-handed, the radiation would be linearly polarized.

To energize each of the helices with equal, in-phase voltages and, at the same time, provide a broad-band transformer between the antennas and a 53-ohm line, the following arrangement is employed. Each antenna is connected by a "single-wire versus ground-plane" transmission line which tapers gradually from about 130 ohms characteristic impedance at the antenna to about 200 ohms at the center of the ground plane. The four lines from the four helices connect in parallel at this point, yielding 50 ohms. The taper from 130 to 200 ohms occurs over a length of about 1 wavelength at the center frequency, so that the transformation is effective over a wide frequency range. The four taper sections are situated on the back side of the ground plane, the helices being on the front. The 53-ohm coaxial line to the transmitter or receiver is introduced at the center of the ground plane from the front, the inner conductor of the coaxial line connecting to the junction point of the four tapered lines.

The ground plane of the antenna which was tested is 94 by 94 cm and the center frequency is 800 Mc. Measured patterns of both the horizontally (H.P.) and vertically (V.P.) polarized components of the radiation are shown in Fig. 10 for frequencies between 600 and

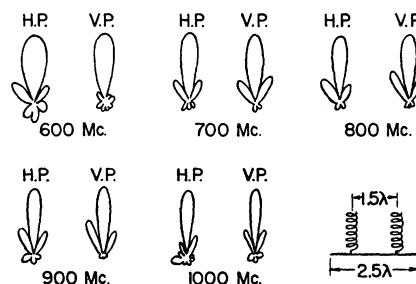


Fig. 10—Measured electric field patterns for 4-helix array shown in Fig. 9.

1000 Mc. All patterns are adjusted to the same maximum value. These patterns agree well with patterns calculated by multiplying the pattern of a single 6-turn 14-degree helix (see Fig. 6) by the array factor for two isotropic point sources separated 1.5 wavelengths at the center frequency. By pattern integration the directivity or gain of the array over an isotropic circularly polarized antenna is about 40 (16 db) at 600 Mc, and about 160 (22 db) at 1000 Mc. These gains are large for an antenna which is 2.5 by 2.5 wavelengths in size at the center frequency. The spacing of 1.5 wavelengths between helices was chosen to provide high gain without regard to side-lobe level. For this arrangement the side-lobe level is determined largely by the level for the single helix.

In Fig. 11 the half-power beam widths for the four-helix array are presented as a function of frequency, as are also curves for the axial ratio in the direction of the helix axis, and the SWR on the 53-ohm transmission

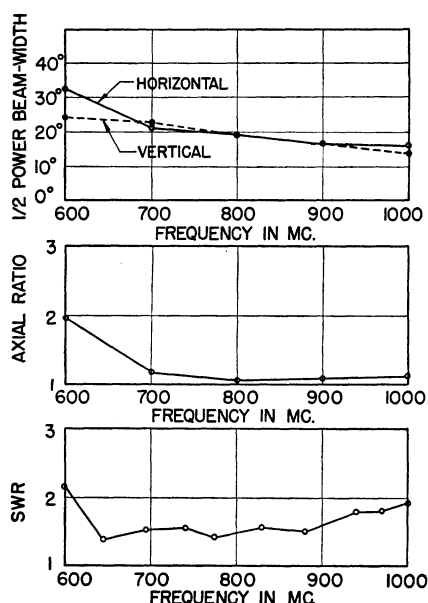


Fig. 11—Summary of measured performance of four-helix array shown in Fig. 9. Half-power beam widths, axial ratio, and standing-wave ratio on a 53-ohm line are presented as a function of frequency.

line. The SWR measurements were made at a distance of about 2.5 meters from the point at which the 53-ohm line connects to the antenna. From an examination of the curves in Fig. 11, all the characteristics of the antenna are satisfactory for operation over most of the 600- to 1000-Mc band, so that the frequency range of the array is nearly as great as for the single 6-turn 14-degree helix.

OMNIDIRECTIONAL ARRAYS USING HELICAL BEAM ANTENNAS

The beam mode of radiation of a helical antenna persists even when the number of turns is reduced to the order of one. Pattern and impedance data for 1-turn helices have been given in footnote references 2 and 3. The patterns may be relatively broad, from 60 to 80 de-

grees between half-power points. The maximum may be in the direction of the helix axis and the axial ratio nearly unity in this direction. However, as with helices of larger n , the axial ratio in general increases in directions away from the axis. Also as indicated in footnote reference 3, the impedance of a single-turn helix is not so constant as when n is 3 or 4 or larger. In spite of these disadvantages, the broad pattern and simplicity of construction of a single-turn helix suggests its application to an omnidirectional circularly polarized antenna.

In Fig. 12 two arrangements are illustrated for an omnidirectional antenna using four helical beam antennas, each of about one turn. The term "omni-

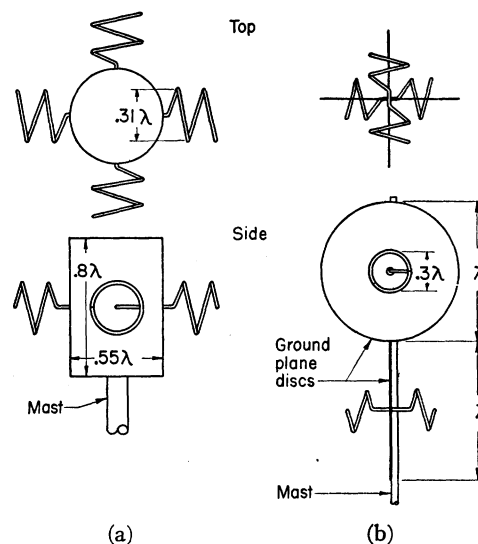


Fig. 12—Two types of omnidirectional helical beam antennas, having four helices of about one turn each. At (a) the helices are disposed around a cylinder. At (b) they are grouped in stacked pairs, with the helices of each pair separated by a flat, circular ground plane.

directional" is used here in the sense of omnidirectional in azimuth only. In Fig. 12(a) (to left) the four helices ($n=1.5$) are arranged around a conducting cylinder about one-half wavelength in diameter at the center frequency. All helices are wound in the same direction and placed on the cylinder in the same orientation. All are energized in phase by transmission lines connected in parallel.

In Fig. 12(b) (to right) the helices are mounted in pairs. Each pair consists of two helices ($n=1$) mounted back-to-back on either side of a circular ground plane one wavelength in diameter. The ground planes are stacked at right angles and spaced one wavelength between centers. Thus, one pair of helices radiates north and south and the other pair east and west. Each pair is connected in parallel. The two pairs are, in turn, connected in parallel and energized from a point midway between the two. The ground-plane diameter and spacing of one wavelength is arbitrary, and smaller values could be used.

In measuring the patterns of these antennas, they were rotated in azimuth (mast as vertical axis) and the field observed with a linearly polarized antenna oriented

successively vertical, horizontal, $+45^\circ$, and -45° . Transmitter power and receiver gain were maintained constant throughout the measurements. The variation of one polarization component (for example, horizontal) usually did not exceed more than about ± 3 db for 360° rotation in azimuth. However, the different polarization components were not, in general, of the same average value, so that the extreme variation of the electric field as a function of both polarization angle and azimuth angle was usually about ± 5 db, but rarely greater. As a function of frequency there appeared to be no marked trend toward either more constant or more irregular patterns over a 1.5 to 1 frequency band. There was also no marked difference between the two types of arrays as regards uniformity of patterns. Although the variation of the electric field of these arrays may be too large for some transmitting applications, the arrays are practical as omnidirectional receiving antennas.

SQUARE HELICAL BEAM ANTENNA FOR SHORT-WAVE USE

A helical beam antenna can be scaled to operate at any frequency. The only limitation is the practical consideration of size. The low-frequency limit may be somewhat reduced by modifying the design to that shown in Fig. 13. The helix is of square cross section¹⁰ and is supported by lines strung between four wooden poles. These lines are broken up by insulators at intervals of a small part of a wavelength. The dimensions given are in free-space wavelengths at the center frequency. The helix shown has 3 turns. A longer helix could be used for greater directivity; for example, one of 6 turns and 14 degrees pitch angle. The spacing between the lower

¹⁰ A helix of square cross section is used in the pattern calculations of footnote reference (2).

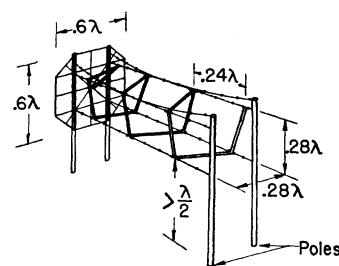


Fig. 13—Helical beam antenna of square cross section for short-wave applications. For operation at 20 Mc (15 meters), four 50-foot poles are required.

side of the helix and the ground should be at least one-half wavelength. A ground plane of spider-web construction is mounted on the far poles. A coaxial transmission line connects the antenna to the transmitting or receiving equipment. It is found that with an antenna of this construction the helix conductor must be sufficiently large (of the order of 0.01 wavelength diameter). The helix conductor may be a large tube (as, for example, stove pipe) or of an open-wire cage construction. By radiating at all polarization angles (circular polarization) this antenna has advantages over linearly polarized types for both transmission and reception. The antenna in Fig. 13 has a gain at the center frequency of more than 10 db over an isotropic circularly polarized radiator.

CONCLUSION

Although helical beam antennas can be applied in other ways, the examples described above illustrate a considerable variety of types and applications.

The author is indebted to Milford C. Horton for his able assistance in the construction and testing of many of the antennas described.

Antenna Design for Television and FM Reception*

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Summary—An approximate method is presented herein of determining, for preliminary design, the resistance and reactance variation with frequency of an antenna or dipole with change of physical dimensions, and to indicate the essential requirements for good performance over a wide band of frequencies necessary for efficient reception of all television channels and FM bands as now allocated for public use by the Federal Communications Commission. A unique antenna system designed to be efficiently responsive over the entire frequency band from 44 to 225 Mc is described.

IN CONSIDERING the simple case of a dipole connected to a transmission line as shown in Fig. 1, it is well, first of all, to examine what happens along the line with variations of the terminal impedance

which the dipole presents at various frequencies. This terminal impedance resolves itself in an equivalent or apparent resistance and reactance in series and is, therefore, a function of the frequency.

A significant measure of the useful range of operating frequencies of an antenna or dipole for reception as well as transmission is the standing-wave ratio along the transmission line resulting from the terminal impedance presented by the antenna or dipole.

The standing-wave ratio is expressed mathematically by the formula,

$$\text{SWR} = \frac{1 + K}{1 - K}$$

where K is the coefficient of reflection, which may, in turn, be mathematically expressed as

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